Physics 380H - Wave Theory — **Homework #10 - Solutions**

Fall 2004 — Due 12:01 PM, Monday 2004/11/29

[45 points total]

“Journal” questions:
– How do you feel about the usefulness and/or effectiveness of these “Journal” types of activities? What do you think their best aspect has been? What change to their format or content might improve their usefulness and/or effectiveness? Why?
– Any comments about this week’s activities? Course content? Assignment? Lab?

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1. (From Towne P14-5, pg 319) Consider an array of N coherent point sources equally spaced along a straight line, the distance between each pair being d. Assume that the sources are of equal strength and in phase.

   (a) Use the Fraunhofer approximation and sum the contributions from each source to determine the net signal at a distant point of observation. (This method of direct summation is equivalent to that used in Towne Chapter 11 for the double source.) [10]

   Solution: The net signal at the distant point of observation will be the sum from al the point sources. As in Towne Chapter 11, since our point is distant, we can assume that the amplitudes $A(r_n)$ are approximately equal to $A(r) = A_0$ where r is the average value of r_n, or equivalently the distance from the centre of the array. We also have that $r_n = r - (n-1)d\sin\theta$ since each source is a distance of d from the next. The phase differences ϕ_n are related to by path differences which are given by $\delta_n = (n-1)d\sin\theta$, namely $\phi_n = k\delta_n = (n-1)kd\sin\theta$.

$$
\begin{aligned}
\psi(P,t) &= \sum_{n=1}^{N}\psi_n(P,t) = \sum_{n=1}^{N} A(r_n)\cos(\omega t - kr_n) \\
&= A_0\sum_{n=1}^{N}\cos(\omega t - kr_n) = A_0\sum_{n=1}^{N}\cos(\omega t - kr + (n-1)kd\sin\theta) \\
&= A_0\sum_{n=1}^{N}\Re\left\{\mathrm{e}^{\mathrm{i}(\omega t - kr + (n-1)kd\sin\theta)}\right\} = A_0\sum_{n=1}^{N}\Re\left\{\mathrm{e}^{\mathrm{i}(\omega t - kr)}\mathrm{e}^{\mathrm{i}(n-1)kd\sin\theta}\right\} \\
&= A_0\Re\left\{\mathrm{e}^{\mathrm{i}(\omega t - kr)}\sum_{n=1}^{N}\mathrm{e}^{\mathrm{i}(n-1)kd\sin\theta}\right\}
\end{aligned}
$$

   This is very reminiscent of the vibration curve construction in Towne Chapter 12, but being a discrete sum rather than an integral. Graphically one could draw out unit vectors head to tail on the complex plane and find their resultant sum in order to get the net wave function. The sum is a *geometric series* of ratio $\mathrm{e}^{-\mathrm{i}(n-1)kd\sin\theta}$, and as in Towne Chapter 14 we have

$$
\begin{aligned}
\sum_{n=1}^{N}\mathrm{e}^{\mathrm{i}(n-1)kd\sin\theta} &= \frac{1-\mathrm{e}^{\mathrm{i}Nkd\sin\theta}}{1-\mathrm{e}^{\mathrm{i}kd\sin\theta}} \\
&= \frac{\mathrm{e}^{-\mathrm{i}(Nkd/2)\sin\theta} - \mathrm{e}^{\mathrm{i}(Nkd/2)\sin\theta}}{\mathrm{e}^{-\mathrm{i}(kd/2)\sin\theta} - \mathrm{e}^{\mathrm{i}(kd/2)\sin\theta}}\cdot\frac{\mathrm{e}^{\mathrm{i}(Nkd/2)\sin\theta}}{\mathrm{e}^{\mathrm{i}(kd/2)\sin\theta}} \\
&= \frac{\sin\left(\frac{Nkd}{2}\sin\theta\right)}{\sin\left(\frac{kd}{2}\sin\theta\right)}\mathrm{e}^{\mathrm{i}(N-1)(kd/2)\sin\theta}.
\end{aligned}
$$

With $\gamma = (kd/2)\sin\theta$ and $\overline{r} = r - (N-1)(d/2)\sin\theta$, we have

$$\begin{aligned}
\psi(P,t) &= A_0 \Re\left\{ \mathrm{e}^{\mathrm{i}(\omega t - k\overline{r})} \left(\frac{\sin N\gamma}{\gamma} \right) \right\} \\
\psi(P,t) &= A_0 \left(\frac{\sin N\gamma}{\gamma} \right) \Re\left\{ \mathrm{e}^{\mathrm{i}(\omega t - k\overline{r})} \right\} \\
\psi(P,t) &= A_0 \left(\frac{\sin N\gamma}{\gamma} \right) \cos\left(\omega t - k\overline{r}\right)
\end{aligned}$$

To find the (average) intensity, we square the wave function (taking the average value of the time varying factor) and multiply by the appropriate constant of proportionality to get

$$I(\theta) = I_0 \left(\frac{\sin N\gamma}{\gamma} \right)^2 .$$

(b) Show that the angular distribution of intensity is the same as that obtained from Towne Equation 14-8 by taking the limit as the length of the individual line segments tends to zero. [5]

Solution: As $a \to 0$, so to does $\beta \to 0$ and $\sin\beta/\beta \to 1$, so Town Equation 14-8 goes to

$$\begin{aligned}
I(\theta) &= I_0 \left(\frac{\sin\beta}{\beta} \right)^2 \left(\frac{\sin N\gamma}{\gamma} \right)^2 = I_0 \left(\frac{\sin\left(\frac{ka}{2}\sin\theta\right)}{\frac{ka}{2}\sin\theta} \right)^2 \left(\frac{\sin N\gamma}{\gamma} \right)^2 \\
I(\theta) &\to I_0 \left(\frac{\sin N\gamma}{\gamma} \right)^2
\end{aligned}$$

Thus as the length of the individual line segments tends to zero, the intensity becomes that due to a N point sources, as expected.

2. (From Towne P14-9, pg 320)

(a) What resolving power is required to resolve the sodium doublet and what information is required about the grating to predict the smallest-order spectrum in which the doublet will be resolved? [5]

Solution: For the sodium doublet $\lambda_1 = 5890\,\text{Å}$ and $\lambda_2 = 5896\,\text{Å}$, so the average wavelength of the doublet is given by $\lambda = (\lambda_1 + \lambda_2)/2 = 5893\,\text{Å}$. The resolving power is given by

$$\frac{\lambda}{(\Delta\lambda)_{\text{min}}} = \frac{(5893\,\text{Å})}{(6\,\text{Å})} = 982.1666\ldots .$$

The resolving power is related to the grating characteristics by

$$\frac{\lambda}{(\Delta\lambda)_{\text{min}}} = nN \quad \Longrightarrow \quad n = \frac{\lambda}{N(\Delta\lambda)_{\text{min}}}$$

where n is the order of the observed spectrum and N is the total number of slits being illuminated. Thus a resolving power of about 982.2 is required to resolve the sodium doublet and in order to predict the smallest-order spectrum in which the doublet will be resolved it is necessary to know how many slits N are being illuminated.

(b) Show that a knowledge of the total width of the grating is sufficient to obtain a lower bound for the *angle* at which a resolved doublet will be obtained. [5]

Solution: The slit spacing d as well as the order and wavelength determine the angle of the observed fringe via

$$d \sin\theta = n\lambda.$$

If we know the total width of the grating l, the slit spacing is a function of the number of lines, namely $d = l/N$. Putting these together we find that the angle of the resolved doublet can be found via

$$\begin{aligned} \sin\theta &= \frac{n\lambda}{d} = \frac{nN\lambda}{l} = \frac{\lambda}{(\Delta\lambda)_{\text{min}}}\frac{\lambda}{l} = \frac{\lambda^2}{l(\Delta\lambda)_{\text{min}}} \\ \theta &= \arcsin\left(\frac{\lambda^2}{l(\Delta\lambda)_{\text{min}}}\right). \end{aligned}$$

Thus, given desired resolving power and the average wavelength, a knowledge of the total width of the grating is sufficient to obtain a lower bound for the angle at which a resolved doublet will be obtained.

(c) Calculate the order number and the angle of the first doublet resolved by a one-inch (2.54 cm) grating of 200 lines. [5]

Solution: For $N = 200$, we would need to use the order $n = 982/200 \approx 5$ to resolve the sodium doublet, this can be found at

$$\begin{aligned} \theta &= \arcsin\left(\frac{n\lambda}{d}\right) == \arcsin\left(\frac{nN\lambda}{l}\right) = \arcsin\left(\frac{(5)(200)(5893\,\text{Å})}{(2.54\,\text{cm})}\right) \\ &= \arcsin(0.02320\ldots) = 0.023202 \approx 0.0232\,\text{radians}. \end{aligned}$$

The doublet is resolved at about 0.0232 radians, corresponding to the 5th order.

3. (From Towne P15-1, pg 372) A string of length l is fixed at $x = 0$ but is "free" at $x = l$. (The device of a frictionless and massless slip ring would be required to maintain tension in the string and yet permit no transverse component of the force acting on the free end.)

   (a) Give arguments to show that the general motion is periodic and deduce the period. [5]

   Solution: At the free end, any waves will be reflected with no phase change, and at the fixed end any waves will be reflected with a phase change of π or $180°$. Thus any arbitrary wave will traverse the length of the string *four* times before it returns to its initial position, travelling in the initial direction. This is in contrast to a string with both ends fixed which has an arbitrary wave returning to the initial position travelling in the initial direction after it travels the length of the string *twice*. Thus for the one-fixed-one-free-end string, the maximum period is $T_{\max} = 4l/c$ and the motion is necessarily periodic.

   Alternatively one could argue that any arbitrary wave can be composed of a sum of sinusoidal waves (via Fourier), and the boundary conditions of $y(0,t) = 0$ (the "fixed" end) and $y'(l,t) = 0$ (the "free" end) give a maximum period of $T_{\max} = 4l/c$ after a bit of math manipulations. Or one could note that the fundamental mode of the string has a wavelength of $4l$, and thus a period of $4l/c$, which is the maximum period of any mode.

   (b) By direct substitution of a function of sinusoidal form determine the frequencies which any sinusoidal solution must have. [5]

   Solution: Let us assume a solution of $y(x,t) = A\sin(kx)\sin(\omega t - \phi)$ which we know is of the proper form for a periodic wave on a string. if we apply the boundary conditions we get the following (note that the first boundary condition of $y(0,t) = 0$ is automatically satisfied by this choice of solution) :

$$[y(x,t)]_{x=0} = 0 \qquad\qquad \left[\frac{\mathrm{d}y(x,t)}{\mathrm{d}x}\right]_{x=l} = 0$$

$$A\sin(k0)\sin(\omega t - \phi) = 0 \qquad\qquad A\omega\cos(kl)\sin(\omega t - \phi) = 0$$

$$\therefore \cos(kl) = 0$$

$$\begin{aligned} kl &= \frac{\pi}{2} + n\pi, \quad n = 0,1,2,3,\ldots \\ k &= \frac{(2n+1)\pi}{2l}, \quad n = 0,1,2,3,\ldots \\ &= \frac{m\pi}{2l}, \quad m = 1,3,5,7,\ldots \end{aligned}$$

   The period is related to k by $T = 2\pi/\omega = 2\pi/kc$ so

$$\begin{aligned} T &= \frac{2\pi}{kc} = \frac{2\pi}{c}\frac{2l}{m\pi}, \quad m = 1,3,5,7,\ldots \\ &= \frac{4l}{mc}, \quad m = 1,3,5,7,\ldots \\ &= \frac{4l}{(2n+1)c}, \quad n = 0,1,2,3,\ldots. \end{aligned}$$

   Thus the maximum value for T is when $m = 1$ (or equivalently $n = 0$) and has a value of $T_{\max} = 4l/c$ which agrees with our arguments above.

(c) Use the method of separation of variables to deduce the necessary form of a product-form solution satisfying the given boundary conditions. [5]

Solution: Following Towne in section 15-4 we assume a solution of product form, namely

$$y(x,t) = X(x)T(t),$$

and apply it to the wave equation to arrive at two differential equations, one for $X(x)$ and one for $T(t)$ by collecting all expressions containing x on one side and those containing t on the other side, and setting them both equal to some constant b. Thus we arrive at

$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{c^2T(t)} = b.$$

The general solution for $X(x)$ is thus

$$X(x) = c_1\mathrm{e}^{\sqrt{b}x} + c_2\mathrm{e}^{-\sqrt{b}x}.$$

Applying our first boundary condition that $y(0,t) = 0$ gives us

$$X(0) = 0 = c_1 + c_2 \quad\Longrightarrow\quad c_2 = -c_1.$$

This is where we deviate from Towne – our second boundary condition is $X'(l,t) = 0$ and along with $c_2 = -c_1$ it gives us

$$\begin{aligned}
\left[\frac{\mathrm{d}X(x)}{\mathrm{d}x}\right]_{x=l} &= \sqrt{b}c_1\mathrm{e}^{\sqrt{b}l} + \sqrt{b}c_1\mathrm{e}^{-\sqrt{b}l} = 0\\
\sqrt{b}c_1\mathrm{e}^{\sqrt{b}l} &= -\sqrt{b}c_1\mathrm{e}^{-\sqrt{b}l}\\
\mathrm{e}^{\sqrt{b}l} &= -\mathrm{e}^{-\sqrt{b}l}\\
\mathrm{e}^{2\sqrt{b}l} &= -1.
\end{aligned}$$

For the complex number $\mathrm{e}^{2\sqrt{b}l}$ to be equal to -1 we must have that $2\sqrt{b}l$ is equal to $\mathrm{i}(\pi + 2n\pi)$, thus

$$\begin{aligned}
2\sqrt{b}l &= \mathrm{i}(\pi + 2n\pi), \quad n = 0,1,2,3,\ldots\\
\sqrt{b} &= \mathrm{i}\frac{(2n+1)\pi}{2l}, \quad n = 0,1,2,3,\ldots\\
\sqrt{b} &= \mathrm{i}\frac{m\pi}{2l}, \quad m = 1,3,5,7,\ldots\\
b &= -\left(\frac{m\pi}{2l}\right)^2, \quad m = 1,3,5,7,\ldots\,.
\end{aligned}$$

With this value for b we can define $k_m = m\pi/(2l)$ (or $k_m = (2n+1)\pi/(2l)$) and we get the expression for $X(x)$ of

$$\begin{aligned}
X(x) &= c_1\left[\mathrm{e}^{\mathrm{i}k_mx} - \mathrm{e}^{-\mathrm{i}k_mx}\right]\\
&= c_3\sin(k_mx), \quad k_m = \frac{m\pi}{2l}, \quad m = 1,3,5,7,\ldots\\
&= c_3\sin(k_nx), \quad k_n = \frac{(2n+1)\pi}{2l}, \quad n = 0,1,2,3,\ldots\,.
\end{aligned}$$

For $T(t)$, we follow Towne exactly and arrive at $T''(t) = -k_m^2c^2T(t) = -\omega_m^2T(t)$ with a general solution of

$$T(t) = c_4\sin(\omega_mt - \phi).$$

Putting it all together we have

$$
\begin{aligned}
y(x,t) &= X(x)T(x) \\
&= C\sin(k_m x)\sin(\omega_m t - \phi), \quad \omega_m = ck_m, \quad k_m = \frac{m\pi}{2l}, \quad m = 1,3,5,7,\ldots \\
&= C\sin(k_n x)\sin(\omega_n t - \phi), \quad \omega_n = ck_n, \quad k_n = \frac{(2n+1)\pi}{2l}, \quad n = 0,1,2,3,\ldots,
\end{aligned}
$$

where ϕ and C are arbitrary constants.

Headstart for next week, Week 11, starting Monday 2004/11/29:
– Read Chapter 15 "Waves Confined to a Limited Region" in Towne, omit 15-14, 15-15
– – Section 15-5 "Linear combination of normal-mode solutions"
– – Section 15-6 "Determination of the coefficients in a normal-modes expansion"
– – Section 15-7 "Independence of the energy contributions from different modes"
– – Section 15-8 "Normal-modes expansion of a rectangular pulse"
– – Section 15-9 "Energy spectrum of the rectangular pulse"
– – Section 15-10 "A too literal interpretation of the normal-modes expansion"
– – Section 15-11 "Normal-modes expansion of a sinusoidal wavetrain of limited extent"