Physics 380H - Wave Theory | **Homework #09 - Solutions**

Fall 2004 | Due 12:01 PM, Monday 2004/11/22

[50 points total]

"Journal" questions:
– Within the subject matter of this course, what do you think the best methods of evaluating student knowledge and/or skills would be? What single change to how we do evaluation in this course do you think would be best? What is the best feature of the evaluation method used in this course? Why?
– Any comments about this week's activities? Course content? Assignment? Lab?

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1. (From Towne P12-5, pg 290) Suppose that the edges of a laboratory slit are opened symmetrically about the centre and are driven at constant velocity. A monochromatic plane wave is incident on the slit and the intensity variations at a fixed point of observation are recorded by means of a photocell. Describe these Fraunhofer radiation intensity variations as a function of time. [10]

   Solution: If the sensor is at a fixed location, while the slit gets bigger, it will have the effect of shrinking the Fraunhofer radiation intensity pattern as time increases. This will have an effect that depends on the location of the sensor. If the sensor is at $\theta = 0$ the intensity will increase monotonically as the size of the slit increases. At other angles close to zero, there may be maxima and minima until the pattern gets so narrow that there is essentially zero intensity at the photosensor's position.

   We have the expression for the Fraunhofer radiation intensity pattern due to a line source of

   $$I(\theta) = I_0 \left(\frac{\sin\beta}{\beta}\right)^2 \qquad \beta = \frac{ka}{2}\sin\theta.$$

   If the slit width is constantly increasing at a speed such that $a = vt$ we can rewrite this expression as

   $$I(\theta) = I_0 \left(\frac{\sin\beta}{\beta}\right)^2 \qquad \beta = \frac{kvt}{2}\sin\theta.$$

   However, this is not the whole picture, since I_0 is actually a function of the slit width too! I_0 is the intensity on the axis $\theta = 0$, and clearly this depends on the size of the slit. As the slit width gets extremely wide, I_0 will tend towards the intensity of the incident plane wave, but for smaller slit widths, the wave function at $\theta = 0$ is proportional to the slit width, and thus the intensity is proportional to the square of the slit width. If we define some new constant of proportionality S as the source intensity per unit length squared (similar to Towne's definition of B in section 12-2) we have $I_0 = Sa^2 = S(vt)^2$ and

   $$\begin{aligned}I(\theta) &= S(vt)^2 \left(\frac{\sin\beta}{\beta}\right)^2 \qquad \beta = \frac{kvt}{2}\sin\theta \\ &= S(vt)^2 \left(\frac{\sin\left(\frac{kvt}{2}\sin\theta\right)}{\frac{kvt}{2}\sin\theta}\right)^2 \\ &= \frac{2S}{k^2}\left(\frac{\sin\left(\frac{kvt}{2}\sin\theta\right)}{\sin\theta}\right)^2.\end{aligned}$$

   We see that this expression is an oscillatory function of t, as t increases for a given angle θ, the intensity goes from zero to max and back with period $T = 2\pi/(kv\sin\theta)$.

2. (From Towne P12-9, pg 290) Consider sound waves generated in water by means of the sinusoidal vibration of a circular piston face of diameter 2 cm. How high must the frequency be to obtain plane waves collimated to within $1°$? [10]

Solution: This problem is equivalent to finding the angle of the Airy disk of a circular aperture. for $a \gg \lambda$ the "conical beam" will have a semiangle such that $\sin\theta \approx \theta = 1.22(\lambda/a)$. To find the minimum frequency we apply the given values and the knowledge that $c = \nu\lambda$ to obtain:

$$\theta = 1.22\frac{\lambda}{a} \quad \Longrightarrow \quad \begin{aligned} \lambda &= \frac{a\theta}{1.22} \\ \frac{c}{\nu} &= \frac{a\theta}{1.22} \\ \nu &= 1.22\frac{c}{a\theta} \end{aligned}$$

Given that for water, the speed of sound is about $c_w = 1483\,\mathrm{m/s}$ and $\theta = 1° = 0.01745$ radians, we have

$$\nu = 1.22\frac{c}{a\theta} = 1.22\frac{(1483\,\mathrm{m/s})}{(0.02\,\mathrm{m})(0.01745\,\mathrm{radians})} = 5.18314\ldots\ \mathrm{MHz} \approx 5.18\,\mathrm{MHz},$$

the frequency of vibration must be at least 5.18 MHz to be collimated to within $1°$.

3. (From Towne P13-2, pg 303) See Towne Fig 13-2

(a) Estimate the values of u_{20} and u_{10} corresponding to the points $\mathbf{Z}_{20}$ and $\mathbf{Z}_{10}$. [5]

Solution: Using a piece of string, or a ruler curving along the spiral, we get a measurement from the origin to point $\mathbf{Z}_{20}$ along the curve of about 10.6 cm. point $\mathbf{Z}_{10}$ is an identical distance, but measured in the negative direction. Measuring the scale of the figure we find that 1.6 units is about 9.62 cm, thus

$$\begin{aligned} u_{20} &= 10.6\,\text{cm}\frac{1.6\,\text{units}}{9.62\,\text{cm}} = 1.7629\,\text{units} \\ &\approx 1.76\,\text{units} \\ u_{10} &\approx -1.76\,\text{units}. \end{aligned}$$

Actually, u_{20} and u_{10} are unitless numbers, so it would be more accurate to say that $u \approx 1.76$.

(b) Consider a slit of fixed width $a = 2\,\text{mm}$ and a wavelength of $\lambda = 5000\,\text{Å}$. At what distance should the observing screen be placed to have the slit edges correspond to $\mathbf{Z}_{10}$ and $\mathbf{Z}_{20}$ when the point of observation is on the axis $\theta = 0$? Sketch $I(\theta)$ for this case. [5]

Solution: Given the slit width and wavelength we can find the value for R which will give the values for u measured above. The arc length along the spiral is given by

$$\begin{aligned} u_{20} - u_{10} &= \sqrt{\frac{2}{\lambda R}}\left(\frac{a}{2} - R\sin\theta\right) - \sqrt{\frac{2}{\lambda R}}\left(-\frac{a}{2} - R\sin\theta\right) \\ &= \sqrt{\frac{2}{\lambda R}}\left(\frac{a}{2} - R\sin\theta + \frac{a}{2} + R\sin\theta\right) \\ &= a\sqrt{\frac{2}{\lambda R}} \\ u_{20} - u_{10} &= a\sqrt{\frac{2}{\lambda R}}, \end{aligned}$$

and is in fact independent of the observation angle. Rearranging we have

$$\begin{aligned} u_{20} - u_{10} = 2u &= a\sqrt{\frac{2}{\lambda R}} \\ 4u^2 &= a^2\frac{2}{\lambda R} \\ R &= \frac{a^2}{\lambda 2u^2} = \frac{(2\,\text{mm})^2}{(5000\,\text{Å})2(1.7629)^2} \\ &= 1.2869\,\text{m} \approx 1.29\,\text{m}. \end{aligned}$$

Thus the screen should be placed at about 1.29 m from the slit. Note that Towne seems to have measured a slightly larger value for u and thus calculated a slightly smaller value for R.

To plot $I(\theta)$ we need to plot

$$\begin{aligned} I(\theta) &= \frac{I_0}{2}\left|\mathbf{Z}_{2-1}\right|^2 \\ &= \frac{I_0}{2}\left|\mathbf{Z}(u_2) - \mathbf{Z}(u_1)\right|^2 \\ &= \frac{I_0}{2}\left[(X(u_2) - X(u_1))^2 + (Y(u_2) - Y(u_1))^2\right] \\ &= \frac{I_0}{2}\left[(C(u_2) - C(u_1))^2 + (S(u_2) - S(u_1))^2\right]. \end{aligned}$$

Since we know the Fresnel integrals $C(c)$ and $S(u)$ as well as u_1 and u_2, we should be able to get Maple to plot $I(\theta)$. As might be expected, only values close to $\theta = 0$ are non-zero (in this case less than about $0.2°$).

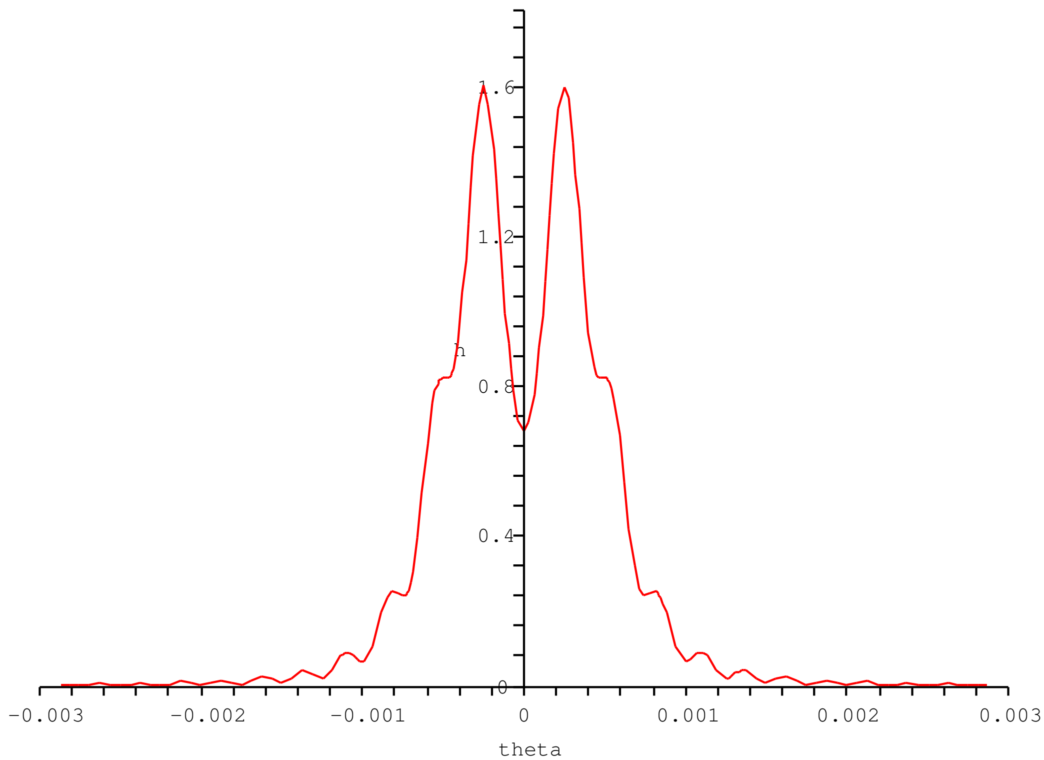

Figure 1: $I_{\text{Fresnel}}(\theta)$ for $R = 1.29\,\text{m}$

We get figure 1 using Maple and the following commands.

```
> restart; with(plots);
> I_0 := 1; lambda := 5000*10^(-10); u_cm := 10.6;
> u := u_cm * 1.6/9.62; a := 0.002; R:=a^2/(lambda*2*u^2);
> u_1 := theta -> sqrt(2/(lambda * R))*(-a/2-R*sin(theta));
> u_2 := theta -> sqrt(2/(lambda * R))*(a/2-R*sin(theta));
> delta_X := theta -> FresnelC(u_2(theta))-FresnelC(u_1(theta));
> delta_Y := theta -> FresnelS(u_2(theta))-FresnelS(u_1(theta));
> I_fresnel := theta -> (I_0/2)*((delta_X(theta))^2 + (delta_Y(theta))^2);
> plot(I_fresnel(theta), theta=(-0.003)..(0.003), h=0..1.8);
```

(c) If the observing screen is moved closer to the slit, at approximately what position will the centre of the pattern be a strong relative maximum? Sketch $I(\theta)$ for this case. [10]

Solution: If the screen is moved closer to the slit, then the arc length $2u$ will become larger, and the points $\mathbf{Z}_{10}$ and $\mathbf{Z}_{20}$ will move further along the spiral, away from the origin. The distance between them, $|\mathbf{Z}_{20-10}|$, will next be a maximum when the points make it around to the outer part of the *second* time around the spirals. This corresponds to a value for u of about 14.1 cm or

$$u = 14.1\,\mathrm{cm}\frac{1.6\,\mathrm{units}}{9.62\,\mathrm{cm}} = 2.34511\,\mathrm{units}.$$

We can calculate R as before

$$\begin{aligned} R &= \frac{a^2}{\lambda 2u^2} = \frac{(2\,\mathrm{mm})^2}{(5000\,\text{Å})2(2.34511)^2} \\ &= 0.727330\,\mathrm{m} \approx 0.73\,\mathrm{m}. \end{aligned}$$

Thus there is a strong relative maximum at a screen distance of about 73 cm. Towne seems to have measured a slightly smaller value for u and thus calculated a slightly larger value for R. Actually, I cheated and used Maple below to try out various values for `u_cm` to get a maximum value for `evalf(I_fresnel(0));` – when I tried to measure carefully I got a distance of about 13.5 cm which gave a value of $R \approx 79\,\mathrm{cm}$, but had a slightly smaller value for `evalf(I_fresnel(0));` than 14.1 cm gave.

As before we can use Maple and the following commands to get figure 2

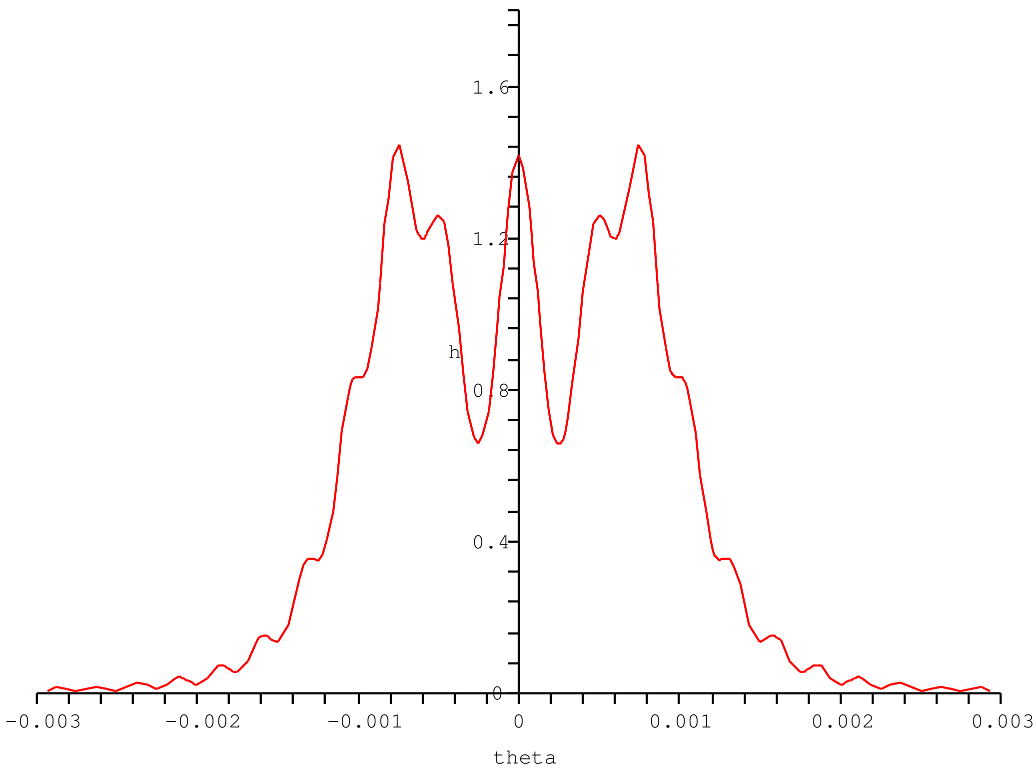

Figure 2: $I_{\text{Fresnel}}(\theta)$ for R at second max

```
> restart; with(plots);
> u_cm := 14.1; u := u_cm * 1.6/9.62; I_0 := 1;
> lambda := 5000*10^(-10); a := 0.002; R:=a^2/(lambda*2*u^2);
> u_1 := theta -> sqrt(2/(lambda * R))*(-a/2-R*sin(theta));
> u_2 := theta -> sqrt(2/(lambda * R))*(a/2-R*sin(theta));
> delta_X := theta -> FresnelC(u_2(theta))-FresnelC(u_1(theta));
> delta_Y := theta -> FresnelS(u_2(theta))-FresnelS(u_1(theta));
> I_fresnel := theta -> (I_0/2)*((delta_X(theta))^2 + (delta_Y(theta))^2);
> plot(I_fresnel(theta), theta=(-0.003)..(0.003), h=0..1.8);
```

4. (From Towne P13-3, pg 304) Let a plane wave of sound be normally incident on a rectangular slit. Assume that the narrow dimension of the slit is sufficiently small so that the diffraction problem can be treated by application of the formulas for the radiation from a coherent linear source. Take the long dimension of the slit to be 10 wavelengths. Consider a microphone which can be moved to different positions along the axis $\theta = 0$ (the perpendicular to the slit at its midpoint).

   (a) According to the Cornu spiral analysis there should be relative maxima and minima as the microphone is moved along the axis. Approximately where are some of these expected to be located? [5]

   Solution: From the previous problem, we know that the second relative maximum is at about $u = 2.35$, making a similar measurement we can find that the first relative maximum is at about 7.2 cm along the spiral, with a value of $u = 1.197$. The first minimum is at about 11.3 cm along the spiral, with a value of $u = 1.88$ and the second minimum is at about 16.5 cm along the spiral, with a value of $u = 2.74$, etc. (I got these values by plotting $I(0)$ in Maple and looking for the relative minima and maxima rather than trying to measure off of the diagram). Now we want to find the R values corresponding to these values of u:

$$R = \frac{a^2}{\lambda 2u^2} = \frac{(10\lambda)^2}{\lambda 2u^2} = \frac{50\lambda}{u^2}$$

$$R_{max-1} = \frac{50\lambda}{(1.197)^2} = (34.9)\lambda \qquad R_{min-1} = \frac{50\lambda}{(1.88)^2} = (14.15)\lambda$$

$$R_{max-2} = \frac{50\lambda}{(2.35)^2} = (9.05)\lambda \qquad R_{min-2} = \frac{50\lambda}{(2.74)^2} = (6.66)\lambda$$

$$R_{max-3} = \frac{50\lambda}{(3.08)^2} = (5.27)\lambda \qquad R_{min-3} = \frac{50\lambda}{(3.39)^2} = (4.35)\lambda.$$

   (b) In what frequency range might this be a feasible experiment? (Consider the limited ability of the microphone to resolve the maxima and minima and the requirement that the dimensions of the slit be reasonable.) [5]

   Solution: The differences between adjacent minima and maxima is on order of only one wavelength, for example $R_{max-3} - R_{min-3} \approx 0.9\lambda$, so our wavelength needs to be about the size of our microphone if the microphone is to be able to measure these positions. For a microphone of about 1 cm in size (larger and smaller ones are available) this gives a maximum frequency of $\nu = c/\lambda = (331\,\mathrm{m/s})/(0.01\,\mathrm{m}) \approx 33\,\mathrm{kHz}$. Since the range of human hearing is only up to about 20 kHz, it seems as though all audible frequencies would be measurable in this type of experiment.

Headstart for next week, Week 10, starting Monday 2004/11/22:
– Read Chapter 14 "The Double Slit; Multiple-slit Arrays; Diffraction Gratings" in Towne
– – Section 14-1 "Introduction"
– – Section 14-2 "The double slit"
– – Section 14-3 "Multiple-slit arrays"
– – Section 14-4 "The diffraction grating"
– Read Chapter 15 "Waves Confined to a Limited Region" in Towne, omit 15-14, 15-15
– – Section 15-1 "Introduction"
– – Section 15-2 "Transverse waves on a string segment with fixed ends"
– – Section 15-3 "Sinusoidal solutions"
– – Section 15-4 "Solutions of product form"