Physics 380H - Wave Theory — Homework #05 - Solutions

Fall 2004 — Due 12:01 PM, Monday 2004/10/18

[50 points total]

"Journal" questions:
– Have you ever noticed any physics (or science or math or technology if you cannot recall a physics example) issue/idea/result presented incorrectly in the general media or popular press? In a non-science course? What was it? What, if anything, should be done about this type of problem? Is it a problem? Why or why not?
– Any comments about this week's activities? Course content? Assignment? Lab?

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1. (From Towne P4-4, pg 81) Calculate the rms values of p, ξ, $\dot{\xi}$, and s in air at standard temperature and pressure for a sinusoidal wave of frequency $\nu = 1000\,\mathrm{sec}^{-1}$ and average intensity $\bar{\imath} = 10^{-12}\,\mathrm{W/m^2}$ [10]

   Solution: For air at one atmosphere and 0°C, $Z = 429\,\mathrm{kg/m^2 s}$. Also

$$\bar{\imath} = p_{\mathrm{rms}}\dot{\xi}_{\mathrm{rms}} = Z\dot{\xi}^2_{\mathrm{rms}} = \frac{p^2_{\mathrm{rms}}}{Z}.$$

$$\begin{aligned}
\Longrightarrow p^2_{\mathrm{rms}} &= Z\bar{\imath} & \dot{\xi}^2_{\mathrm{rms}} &= \frac{\bar{\imath}}{Z} \\
p_{\mathrm{rms}} &= \sqrt{Z\bar{\imath}} & \dot{\xi}_{\mathrm{rms}} &= \sqrt{\frac{\bar{\imath}}{Z}} \\
&= \sqrt{(429\,\mathrm{kg/m^2s})(10^{-12}\,\mathrm{W/m^2})} & &= \sqrt{\frac{(10^{-12}\,\mathrm{W/m^2})}{(429\,\mathrm{kg/m^2s})}} \\
&= \sqrt{(4.29\times10^{-10}\,\mathrm{W\cdot kg/m^4s})} & &= \sqrt{(2.33100\times10^{-15}\,\mathrm{W\cdot s/kg})} \\
&= (2.071231\ldots\times10^{-5}\,\mathrm{N/m^2}) & &= (4.8280\ldots\times10^{-8}\,\mathrm{m/s}) \\
p_{\mathrm{rms}} &\approx (2.07\times10^{-5}\,\mathrm{N/m^2}) & \dot{\xi}_{\mathrm{rms}} &\approx (4.83\times10^{-8}\,\mathrm{m/s})
\end{aligned}$$

   The value for $\mathcal{B}_a$ for air can be calculated by $\mathcal{B}_a = \rho_0 c^2$ and since $\rho_0 = 1.293\,\mathrm{kg/m^3}$ and $c = 331\,\mathrm{m/s}$ for air we have

$$\begin{aligned}
p_{\mathrm{rms}} = \mathcal{B}_a s_{\mathrm{rms}} \quad \Longrightarrow \quad s_{\mathrm{rms}} &= \frac{p_{\mathrm{rms}}}{\mathcal{B}_a} = \frac{p_{\mathrm{rms}}}{\rho_0 c^2} = \frac{\sqrt{Z\bar{\imath}}}{\rho_0 c^2} \\
&= \frac{\sqrt{(429\,\mathrm{kg/m^2s})(10^{-12}\,\mathrm{W/m^2})}}{(1.293\,\mathrm{kg/m^3})(331\,\mathrm{m/s})^2} \\
&= \frac{\sqrt{(429\,\mathrm{kg/m^2s})(10^{-12}\,\mathrm{W/m^2})}}{(1.293\,\mathrm{kg/m^3})(331\,\mathrm{m/s})^2} \\
&= (1.46209\ldots\times10^{-10}) \\
s_{\mathrm{rms}} &\approx (1.46\times10^{-10})
\end{aligned}$$

   In order to calculate $\dot{\xi}_{\mathrm{rms}}$, we need to know the relationship between the amplitude of ξ and $\dot{\xi}$ for a sinusoidal wave, since the rms values will have the same relationship. We can write ξ as

$$\begin{aligned}
\xi = \xi_m \sin(kx - \omega t) \quad \Longrightarrow \quad \dot{\xi} &= \frac{\partial\xi}{\partial t} = \xi_m \frac{\partial}{\partial t}\sin(kx-\omega t) \\
&= -\xi_m\omega\cos(kx-\omega t) \\
&= \dot{\xi}_m\cos(kx-\omega t)
\end{aligned}$$

$$\therefore \left|\dot{\xi}_m\right| = |\xi_m\omega| \quad \Longrightarrow \quad \dot{\xi}_{\mathrm{rms}} = \xi_{\mathrm{rms}}\omega$$

We know however that $\omega = 2\pi\nu$, so

$$\begin{aligned}
\therefore \xi_{\text{rms}} &= \frac{\dot{\xi}_{\text{rms}}}{2\pi\nu} = \frac{1}{2\pi\nu}\sqrt{\frac{\bar{\imath}}{Z}} \\
&= \frac{1}{2\pi(1000\,\text{s}^{-1})}\sqrt{\frac{(10^{-12}\,\text{W/m}^2)}{(429\,\text{kg/m}^2\text{s})}} \\
&= (7.68407\ldots \times 10^{-12}\,\text{m}) \\
\xi_{\text{rms}} &\approx (7.68 \times 10^{-12}\,\text{m})
\end{aligned}$$

2. (From Towne P4-6. pg 82)Assume that the displacement amplitude of a vibrating piston is independent of the medium in which it is operating.

   (a) Compare the power outputs of the piston in water and air. [5]

   Solution: The radiative intensity is what we are interested in here, since that will give us a measure of the power per unit area. The average radiative intensity is a function of p_m and $\dot{\xi}_m$, the pressure and displacement velocity amplitudes by way of:

$$\bar{\imath} = \frac{p_m\dot{\xi}_m}{2} = \frac{Z\dot{\xi}_m^2}{2} = \frac{p_m^2}{2Z}.$$

   We could use any of these expressions in addition to the knowledge that ξ_m is the same for water and air, and also the assumption that the piston frequency is the same in both water and air. We want to find the ratio of $\bar{\imath}_{\text{water}}$ and $\bar{\imath}_{\text{air}}$, so we will need to find expressions for $\dot{\xi}_m$ and/or p_m in terms of the physical parameters of the system, with the knowledge that $\mathcal{B}_a = \rho_0 c^2 = Zc$. For a sinusoidal wave we have

$$\begin{aligned}
\xi = \xi_m \sin(kx - \omega t) \quad \implies \quad \dot{\xi} &= \frac{\partial \xi}{\partial t} = \xi_m \frac{\partial}{\partial t}\sin(kx - \omega t) \\
&= -\xi_m \omega \cos(kx - \omega t) \\
&= \dot{\xi}_m \cos(kx - \omega t) \\
\therefore \left|\dot{\xi}_m\right| &= |\xi_m \omega|,
\end{aligned}$$

   in both water and air, thus we can calculate the ratio of interest by

$$\begin{aligned}
\frac{\bar{\imath}_{\text{water}}}{\bar{\imath}_{\text{air}}} &= \frac{Z_{\text{water}}\dot{\xi}_m^2}{2}\frac{2}{Z_{\text{air}}\dot{\xi}_m^2} \\
&= \frac{Z_{\text{water}}}{Z_{\text{air}}} = \frac{\rho_{\text{water}}c_{\text{water}}}{\rho_{\text{air}}c_{\text{air}}} = Z_{\text{wa}} = \frac{1}{Z_{\text{aw}}} \\
&= \frac{(1480000)}{(429)} = \frac{(998)(1483)}{(1.293)(331)} \\
&\approx 3450.
\end{aligned}$$

   Thus the piston delivers about 3450 times as much power to the water than to the air.

(b) If the piston is under water and parallel to a water-air surface, compare the intensity of the wave transmitted into the air with the intensity of the wave obtained when the piston is operating directly in air. [5]

Solution: Here we want to compare $\bar{\imath}_{\text{air}}$ with the intensity of a wave that starts in water and is then transmitted into the air, which we could call $\bar{\imath}_{\text{water}\to\text{air}}$. To find $\bar{\imath}_{\text{water}\to\text{air}}$ we need to know the transmission coefficient T_i between water and air. Since Z_{air} is so much smaller than Z_{water}, the relative impedance will be much less than one, so we can use the approximations

$$Z_{\text{aw}} = \frac{Z_{\text{air}}}{Z_{\text{water}}} \approx \frac{1}{3450} \quad \Longrightarrow \quad T_p \approx 2 - 2Z_{\text{aw}} \qquad T_{\dot{\xi}} \approx 2Z_{\text{aw}}$$

$$\begin{aligned} T_i &= T_p T_{\dot{\xi}} \\ &\approx (2 - 2Z_{\text{aw}})(2Z_{\text{aw}}) \\ &= 4Z_{\text{aw}} - 4Z_{\text{aw}}^2 \\ &\approx 4Z_{\text{aw}} \end{aligned}$$

So the intensity $\bar{\imath}_{\text{water}\to\text{air}}$ can be calculated in terms of the results of the first part of the problem

$$\begin{aligned} \bar{\imath}_{\text{water}\to\text{air}} &= \bar{\imath}_{\text{water}} T_i \\ &= \bar{\imath}_{\text{water}} 4Z_{\text{aw}} \\ &= \frac{\bar{\imath}_{\text{air}}}{Z_{\text{aw}}} 4Z_{\text{aw}} \\ &= 4\bar{\imath}_{\text{air}} \end{aligned}$$

Thus while the transmitted wave in the air is $4Z_{\text{aw}} \approx 0.00116$ of the wave produced by the piston in the water, it is still four times greater than the wave that would have been produced by the piston in air.

3. (From Towne P4-9, pg 82) A room having a volume of $1000\,\mathrm{m}^3$ is filled with a sound wave of intensity level 60 db.

   (a) Estimate the total energy present. [5]

   Solution: The intensity level in decibels is given, so we can calculate the sound intensity by

$$\begin{aligned}\Delta &= 10\log_{10}\left(\frac{i}{i_0}\right) \quad\Longrightarrow\quad \frac{\Delta}{10} = \log_{10}\left(\frac{i}{i_0}\right) \quad\Longrightarrow\quad \frac{i}{i_0} = 10^{\frac{\Delta}{10}} \\ \therefore i &= i_0\left(10^{\frac{\Delta}{10}}\right) \\ &= 10^{-12}\,\mathrm{W/m^2}\left(10^{\frac{60}{10}}\right) = 10^{-6}\,\mathrm{W/m^2}.\end{aligned}$$

   The potential and kinetic energy densities are given by

$$w_{tot} = w_{kin} + w_{pot} = \frac{1}{2}\rho_0\dot{\xi}^2 + \frac{p^2}{2\mathcal{B}_a}$$

   where for a progressive sinusoidal wave, the potential and kinetic terms are equal. Since we also know the intensity relationships for such a wave, we can for example calculate p^2 and $\dot{\xi}^2$ in terms of i

$$\begin{aligned}i = p\dot{\xi} = Z\dot{\xi}^2 &= \frac{p^2}{Z} \quad\Longrightarrow\quad p^2 = Zi, \qquad \dot{\xi}^2 = \frac{i}{Z} \\ w_{tot} &= \frac{1}{2}\rho_0\dot{\xi}^2 + \frac{p^2}{2\mathcal{B}_a} \\ &= \rho_0\frac{i}{2Z} + \frac{Zi}{2\mathcal{B}_a} = \frac{i\rho_0}{2\rho_0 c} + \frac{Zi}{2Zc} \\ &= \frac{i}{2c} + \frac{i}{2c} = \frac{i}{c}\end{aligned}$$

   Since this is an energy density, the total energy is calculated by multiplying by the volume:

$$\begin{aligned}E_{tot} = Vw_{tot} &= \frac{Vi}{c} \\ &= \frac{(1000\,\mathrm{m}^3)(10^{-6}\,\mathrm{W/m^2})}{(331\,\mathrm{m/s})} \\ &= 3.0211\times10^{-6}\,\mathrm{J} \\ &\approx 3\times10^{-6}\,\mathrm{J}\end{aligned}$$

   (b) At what intensity level would a total energy of 1 calorie be achieved? [5]

   Solution: For a given total energy $E_{tot} = 1\,\mathrm{cal} = 4.186\,\mathrm{J}$ we have:

$$\begin{aligned}E_{tot} &= \frac{Vi}{c} \quad\Longrightarrow\quad i = \frac{E_{tot}c}{V} \\ \Delta &= 10\log_{10}\left(\frac{i}{i_0}\right) = 10\log_{10}\left(\frac{E_{tot}c}{Vi_0}\right) \\ &= 10\log_{10}\left(\frac{(4.186\,\mathrm{J})(331\,\mathrm{m/s})}{(1000\,\mathrm{m}^3)(10^{-12}\,\mathrm{W/m^2})}\right) \\ &= 10\log_{10}\left(1.3855\times10^{12}\right) \approx 121.4\,\mathrm{db}\end{aligned}$$

   So 1 calorie of sound energy in a room of this size is about 121 db loud, which is well into the pain levels for unprotected ears.

4. (From Towne P6-4, pg 109) Show that Maxwell's equations permit a solution in which all the components of **E** and **H** vanish identically everywhere except for: [10]

   (a) E_z and H_y

   (b) E_x and H_z

   Describe the situation represented by each of these solutions.

   Solution: This follows directly from the procedure used in Towne, Chapter 6. Maxwell's equations in free space are:
   $\nabla \cdot \mathbf{E} = 0,$ $\nabla \cdot \mathbf{H} = 0,$ $\nabla \times \mathbf{E} = -\mu(\partial \mathbf{H}/\partial t),$ $\nabla \times \mathbf{H} = \epsilon(\partial \mathbf{E}/\partial t).$

   If all the components of **E** and **H** vanish identically everywhere except for E_z and H_y, Maxwell's equations become

$$\frac{\partial E_z}{\partial z} = 0 \tag{4.01}$$

$$\frac{\partial H_y}{\partial y} = 0 \tag{4.02}$$

$$\frac{\partial E_z}{\partial y}\hat{\imath} - \frac{\partial E_z}{\partial x}\hat{\jmath} = -\mu\frac{\partial H_y}{\partial t}\hat{\jmath} \tag{4.03}$$

$$-\frac{\partial H_y}{\partial z}\hat{\imath} + \frac{\partial H_y}{\partial x}\hat{k} = \epsilon\frac{\partial E_z}{\partial t}\hat{k}. \tag{4.04}$$

   Taking the separate components of (4.03) and (4.04) we get

$$\frac{\partial E_z}{\partial y} = 0 \tag{4.05}$$

$$\frac{\partial E_z}{\partial x} = \mu\frac{\partial H_y}{\partial t} \tag{4.06}$$

$$\frac{\partial H_y}{\partial z} = 0 \tag{4.07}$$

$$\frac{\partial H_y}{\partial x} = \epsilon\frac{\partial E_z}{\partial t}. \tag{4.08}$$

   The equations (4.01), (4.02), (4.05) and (4.07) will only be satisfied if E_z and H_y are functions of at most x and t. If we differentiate (4.06) with respect to x and (4.08) with respect to t we get

$$\frac{\partial^2 E_z}{\partial x^2} = \mu\frac{\partial^2 H_y}{\partial x \partial t} \tag{4.09}$$

$$\frac{\partial^2 H_y}{\partial t \partial x} = \epsilon\frac{\partial^2 E_z}{\partial t^2}. \tag{4.10}$$

   Since the order of differentiating (t or x) does not matter, we can put (4.10) into (4.09) to get

$$\frac{\partial^2 E_z}{\partial x^2} = \mu\epsilon\frac{\partial^2 E_z}{\partial t^2}. \tag{4.11}$$

   Similarly, we can differentiate (4.06) with respect to t and (4.08) with respect to x and put the results together to arrive at

$$\frac{\partial^2 E_z}{\partial t \partial x} = \mu\frac{\partial^2 H_y}{\partial t^2}$$

$$\frac{\partial^2 H_y}{\partial x^2} = \epsilon\frac{\partial^2 E_z}{\partial x \partial t}$$

$$\implies \frac{\partial^2 H_y}{\partial x^2} = \epsilon\mu\frac{\partial^2 H_y}{\partial t^2}. \tag{4.12}$$

Equations (4.11) and (4.12) are wave equations with $c^2 = 1/\mu\epsilon$ and we have shown that Maxwell's equations do permit a solution in which all the components of **E** and **H** vanish identically everywhere except for $E_z(x,t)$ and $H_y(x,t)$.

This situation is one where the electric and magnetic waves propagate in either direction along the $|\hat{k} \times \hat{\jmath}| = \hat{\imath}$, or x, direction with the electric field only having z components, and the magnetic field only having y components. The direction of propagation not only follows from the direction of $\mathbf{S} = \mathbf{E} \times \mathbf{H}$, but also from the fact that both functions are only functions of only x and t. Without knowing the functional form of the x and t dependance for **E** and **H**, we cannot tell if the waves are going in the positive, negative, or both directions.

If all the components of **E** and **H** vanish identically everywhere except for E_x and H_z, we follow the same procedure and, Maxwell's equations become

$$\frac{\partial E_x}{\partial x} = 0 \tag{4.13}$$

$$\frac{\partial H_z}{\partial y} = 0 \tag{4.14}$$

$$\frac{\partial E_x}{\partial z}\hat{\jmath} - \frac{\partial E_x}{\partial y}\hat{k} = -\mu\frac{\partial H_z}{\partial t}\hat{k} \tag{4.15}$$

$$\frac{\partial H_z}{\partial y}\hat{\imath} - \frac{\partial H_z}{\partial x}\hat{\jmath} = \epsilon\frac{\partial E_x}{\partial t}\hat{\imath}. \tag{4.16}$$

Taking the separate components of (4.15) and (4.16) we get

$$\frac{\partial E_x}{\partial z} = 0 \tag{4.17}$$

$$\frac{\partial E_x}{\partial y} = \mu\frac{\partial H_z}{\partial t} \tag{4.18}$$

$$\frac{\partial H_z}{\partial x} = 0 \tag{4.19}$$

$$\frac{\partial H_z}{\partial y} = \epsilon\frac{\partial E_x}{\partial t}. \tag{4.20}$$

The equations (4.13), (4.14), (4.17) and (4.19) will only be satisfied if E_x and H_z are functions of at most y and t. If we differentiate (4.18) with respect to y and (4.20) with respect to t we get

$$\frac{\partial^2 E_x}{\partial y^2} = \mu\frac{\partial^2 H_z}{\partial y \partial t} \tag{4.21}$$

$$\frac{\partial^2 H_z}{\partial t \partial y} = \epsilon\frac{\partial^2 E_x}{\partial t^2}. \tag{4.22}$$

Since the order of differentiating (t or y) does not matter, we can put (4.22) into (4.21) to get

$$\frac{\partial^2 E_x}{\partial y^2} = \mu\epsilon\frac{\partial^2 E_x}{\partial t^2}. \tag{4.23}$$

Similarly, we can differentiate (4.18) with respect to t and (4.20) with respect to y and put the results together to arrive at

$$\frac{\partial^2 E_x}{\partial t \partial x} = \mu\frac{\partial^2 H_z}{\partial t^2}$$

$$\frac{\partial^2 H_z}{\partial y^2} = \epsilon\frac{\partial^2 E_x}{\partial y \partial t}$$

$$\Longrightarrow \frac{\partial^2 H_z}{\partial y^2} = \epsilon\mu\frac{\partial^2 H_z}{\partial t^2}. \tag{4.24}$$

Equations (4.23) and (4.24) are wave equations with $c^2 = 1/\mu\epsilon$ and we have shown that Maxwell's equations do permit a solution in which all the components of $\mathbf{E}$ and $\mathbf{H}$ vanish identically everywhere except for $E_x(y,t)$ and $H_z(y,t)$.

This situation is one where the electric and magnetic waves propagate in either direction along the $|\hat{\imath} \times \hat{k}| = \hat{\jmath}$, or y, direction with the electric field only having x components, and the magnetic field only having z components. The direction of propagation not only follows from the direction of $\mathbf{S} = \mathbf{E} \times \mathbf{H}$, but also from the fact that both functions are only functions of only y and t. Without knowing the functional form of the y and t dependance for $\mathbf{E}$ and $\mathbf{H}$, we cannot tell if the waves are going in the positive, negative, or both directions.

5. (From Towne P7-4, pg 131) In optical systems which involve lenses, a loss of intensity is encountered due to reflection at the lens surfaces. Assume a relative index of refraction of 1.5 and calculate the percent loss in intensity which occurs at each passage from air to glass or glass to air. (Note: The theory of image formation by a lens assumes that all rays are nearly parallel to the axis of the lens. Consequently it is justified to assume normal incidence in this problem.) [10]

   Solution: We are given $n_{12} = 1.5$, for normal incidence we know that

$$
\begin{aligned}
R_E &= \frac{1 - n_{12}}{1 + n_{12}} \\
&= \frac{1 - 1.5}{1 + 1.5} = \frac{-0.5}{2.5} = -0.2.
\end{aligned}
$$

   This is the reflection coefficient for the amplitude of the electric field. Really we want to know R_S, the reflection coefficient for the intensity (derived in Towne from the isomorphism to the acoustic waves), namely

$$
R_S = -R_E^2 = -\left(\frac{1 - n_{12}}{1 + n_{12}}\right)^2 = -(-0.2)^2 = -0.4
$$

   Since we are interested in only the amplitude (not the phase) we can take the absolute value $|R_S|$ to find that the percentage loss in intensity due to reflection is 4% at each interface.

Headstart for next week, Week 06, starting Monday 2004/10/18:
– Mid term test Friday October 22, up to and including material from Chapter 6.
– Read Chapter 7 "Analytical Description of Polarized Electromagnetic Plane Waves" in Towne
– – Section 7-6 "Types of polarization"
– – Section 7-7 "Natural light"
– – Section 7-8 "Energy relations for the general progressive plane wave"
– – Section 7-9 "Reflections by a thin film"