Physics 380H - Wave Theory | **Homework #02 - Solutions**

Fall 2004 | Due 12:01 PM, Monday 2004/09/27

[50 points total]

"Journal" questions:
– What is your major/minor/etc.? What are you planning on doing after you finish your degree?
– Any comments about this week's activities? Course content? Assignment? Lab?

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1. (From Towne P1-2. pg 17) If $h(u)$ is an arbitrary twice-differentiable function of u, show by direct calculation that $y(x,t) = h(t + x/c)$ satisfies the one-dimensional wave equation. What relation does this solution have to the most general form of solution to the one-dimensional wave equation $y(x,t) = f(x - ct) + g(x + ct)$? [10]

   Solution: Given $y(x,t) = h(t + x/c)$, we want to show that y is a solution to the wave equation:
   $$\frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2}.$$
   Thus we need to do the partial derivatives:
   $$\frac{\partial h}{\partial t} = h'\frac{\partial(t + x/c)}{\partial t} = h'(t + x/c)$$
   $$\frac{\partial^2 h}{\partial t^2} = \frac{\partial}{\partial t}\left[h'(t + x/c)\right] = h''\frac{\partial(t + x/c)}{\partial t} = h''(t + x/c)$$
   $$\frac{\partial h}{\partial x} = h'\frac{\partial(t + x/c)}{\partial t} = \frac{1}{c}h'(t + x/c)$$
   $$\frac{\partial^2 h}{\partial t^2} = \frac{\partial}{\partial t}\left[\frac{h'(t + x/c)}{c}\right] = \frac{h''}{c}\frac{\partial(t + x/c)}{\partial t} = \frac{1}{c^2}h''(t + x/c)$$
   From this we see:
   $$\frac{1}{c^2}\frac{\partial^2 h}{\partial t^2} = \frac{1}{c^2}h'(t + x/c) = \frac{\partial^2 h}{\partial t^2}$$
   $$\frac{1}{c^2}\frac{\partial^2 h}{\partial t^2} = \frac{\partial^2 h}{\partial t^2}$$
   which is the wave equation. Thus $y(x,t) = h(t + x/c)$ satisfies the one-dimensional wave equation.

   $y(x,t) = h(t + x/c)$ is related to the most general form of solution to the one-dimensional wave equation $y(x,t) = f(x - ct) + g(x + ct)$ through the function $g(x + ct)$, which represents the waves moving to the left, in the negative x direction. This can be seen by slightly re-writing the function h:
   $$y(x,t) = h\left(t + \frac{x}{c}\right) = h\left(\frac{c}{c}t + \frac{1}{c}x\right) = h\left(\frac{1}{c}\left[ct + x\right]\right) = h\left(\frac{1}{c}\left[x + ct\right]\right)$$
   With this manipulation we can define a new function, say g_1 which is a function of $x + ct$, thus taking the same form as g:
   $$g_1(x + ct) \equiv h\left(\frac{1}{c}\left[x + ct\right]\right),$$
   which is a wave moving to the left, in the negative x direction.

2. (From Towne PI-2. pg 436) In general, the sum of two sinusoidal functions of equal amplitude but different phase has an amplitude different from that of the original functions. Find the exception. That is, find the value of θ which will allow $A\cos x + A\cos(x+\theta) = A\cos(x+\theta')$. Find also θ', the phase of the resultant. [10]

Solution: One of the important points to recognize is that this relationship needs to hold for *all* values of x. Since we have two "unknowns", namely θ and θ', we will need at least two relations to figure out their values. Using two different values of x should be sufficient. Any two values in theory should work, but the easiest ones would be values that make the resulting relations as simple as possible. One value of possible use would be $x = 0$ since that simplifies each term quite a bit. Another useful value would be $x = \pm\pi/2$ since shifting a cosine function by one quarter of a phase would turn it into a $\pm$sine function. So we then get:

$$\begin{aligned}
x = 0 \quad \Longrightarrow \quad A\cos 0 + A\cos(0+\theta) &= A\cos\left(0+\theta'\right) \\
1 + \cos(\theta) &= \cos\left(\theta'\right) \\
\cos\left(\theta'\right) - \cos(\theta) &= 1 \\
2\sin\left(\frac{\theta+\theta'}{2}\right)\sin\left(\frac{\theta-\theta'}{2}\right) &= 1 \\
\sin\left(\frac{\theta+\theta'}{2}\right)\sin\left(\frac{\theta-\theta'}{2}\right) &= \frac{1}{2} \qquad (1)
\end{aligned}$$

$$\begin{aligned}
x = \frac{\pi}{2} \quad \Longrightarrow \quad A\cos\frac{\pi}{2} + A\cos\left(\frac{\pi}{2}+\theta\right) &= A\cos\left(\frac{\pi}{2}+\theta'\right) \\
0 + \sin(\theta) &= \sin\left(\theta'\right) \\
\sin\left(\theta'\right) - \sin(\theta) &= 0 \\
\cos\left(\frac{\theta+\theta'}{2}\right)\sin\left(\frac{\theta-\theta'}{2}\right) &= 0 \qquad (2)
\end{aligned}$$

Comparing (1) and (2) (for example by dividing (2) by (1)) we see that:

$$\cos\left(\frac{\theta+\theta'}{2}\right) = 0 \quad \Longrightarrow \quad \left(\frac{\theta+\theta'}{2}\right) = \pi\left(n+\frac{1}{2}\right) \quad \Longrightarrow \quad \sin\left(\frac{\theta+\theta'}{2}\right) = 1 \qquad (3)$$

Putting (3) into (1) gives us:

$$\sin\left(\frac{\theta-\theta'}{2}\right) = \frac{1}{2} \quad \Longrightarrow \quad \left(\frac{\theta-\theta'}{2}\right) = \frac{\pi}{6} + n'\pi$$

For the simplest solutions, we will take $n = n' = 0$ so

$$\left(\frac{\theta+\theta'}{2}\right) = \frac{\pi}{2} \qquad (4)$$

$$\left(\frac{\theta-\theta'}{2}\right) = \frac{\pi}{6} \qquad (5)$$

We can find the sum and difference of (4) and (5) to get the results that we are looking for:

$$\begin{aligned}
(4) + (5) \quad &\Longrightarrow \quad \theta = \frac{\pi}{2} + \frac{\pi}{6} \quad &\Longrightarrow \quad \theta = \frac{2\pi}{3} \\
(4) - (5) \quad &\Longrightarrow \quad \theta' = \frac{\pi}{2} - \frac{\pi}{6} \quad &\Longrightarrow \quad \theta' = \frac{\pi}{3}
\end{aligned}$$

3. (From Towne Appendix I. pg 433) Prove the identity: $\cos\alpha - \cos\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\beta-\alpha}{2}\right)$, by using the representation of sin and cos as complex functions. Of possible use are the identities: $e^{i\theta} = \cos\theta + i\sin\theta$, $\cos\theta = \Re e^{i\theta} = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$, and $\sin\theta = \Im e^{i\theta} = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$. [10]

   Solution: One applies the identities and turns the crank. It is perhaps easier to start on the RHS and work towards the LHS, but either way will do:

$$
\begin{aligned}
\cos\alpha - \cos\beta &= \frac{1}{2}\left(e^{i\alpha} + e^{-i\alpha}\right) - \frac{1}{2}\left(e^{i\beta} + e^{-i\beta}\right) \\
&= \frac{1}{-2}\left(-e^{i\alpha} - e^{-i\alpha} + e^{i\beta} + e^{-i\beta}\right) \\
&= \frac{1}{-2}\left(-e^{+\frac{i\alpha}{2}+\frac{i\alpha}{2}} - e^{-\frac{i\alpha}{2}-\frac{i\alpha}{2}} + e^{+\frac{i\beta}{2}+\frac{i\beta}{2}} + e^{-\frac{i\beta}{2}-\frac{i\beta}{2}}\right) \\
&= \frac{1}{2i^2}\left(-e^{\frac{+i\alpha}{2}}e^{\frac{+i\alpha}{2}} - e^{\frac{-i\alpha}{2}}e^{\frac{-i\alpha}{2}} + e^{\frac{+i\beta}{2}}e^{\frac{+i\beta}{2}} + e^{\frac{-i\beta}{2}}e^{\frac{-i\beta}{2}}\right) \\
&= \frac{2}{4i^2}\left(e^{\frac{+i\alpha}{2}}e^{\frac{+i\beta}{2}} - e^{\frac{-i\alpha}{2}}e^{\frac{-i\beta}{2}}\right)\left(e^{\frac{+i\beta}{2}}e^{\frac{-i\alpha}{2}} - e^{\frac{+i\alpha}{2}}e^{\frac{-i\beta}{2}}\right) \\
&= 2\left[\frac{1}{2i}\left(e^{\frac{i(\alpha+\beta)}{2}} - e^{\frac{-i(\alpha+\beta)}{2}}\right)\frac{1}{2i}\left(e^{\frac{i(\beta-\alpha)}{2}} - e^{\frac{-i(\beta-\alpha)}{2}}\right)\right] \\
&= 2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\beta-\alpha}{2}\right)
\end{aligned}
$$

4. (From Towne P2-3. pg 36) When a tin flute is attached to a source of illuminating gas it is found that the pitch is approximately a musical third lower than the corresponding pitch in air. What does this mean in terms of the frequency of the sound? Estimate the molecular weight of the gas. [10]

   Solution: If the pitch is "approximately a musical third lower" then the frequency has been decreased by a factor of 4/5 (see for example http://hypertextbook.com/physics/waves/music/), thus

$$\nu_{gas} = \nu_{air}\frac{4}{5}.$$

   The length of the flute does not change, so the wavelength of the sound wave λ remains unchanged, while the speed of sound c is altered since $c = \nu/\lambda$.

$$\nu_{gas} = c_{gas}\lambda = \nu_{air}\frac{4}{5} = c_{air}\lambda\frac{4}{5}. \quad \Longrightarrow \quad c_{gas} = c_{air}\frac{4}{5} \quad \Longrightarrow \quad \frac{c_{gas}}{c_{air}} = \frac{4}{5}.$$

   We know the velocity of sound is given by:

$$c_{gas}^2 = \frac{\gamma R T_o}{M_{gas}} \quad , \quad c_{air}^2 = \frac{\gamma R T_o}{M_{air}} \quad \Longrightarrow \quad \frac{c_{gas}^2}{c_{air}^2} = \frac{M_{air}}{M_{gas}} = \frac{16}{25} = 0.64,$$

   assuming γ is about the same for each gas. This gives us the final result of

$$M_{gas} = \frac{25 M_{air}}{16}.$$

   Since the molecular mass (often called the molecular weight) of air is about 28.9 g/mol (see for example http://members.axion.net/~enrique/molecularweight.html, and others), the molecular weight of our "mystery gas" is a bit less than one and a half of this value, or about: 45.2 g/mol.

5. (From Towne P2-5. pg 36) An organ pipe is tuned to a pitch of 440 Hz when the temperature is 25°C. What will the pitch be at a temperature of 0°C? [10]

Solution: Given:

$$T_1 = 25^\circ\,\mathrm{C}$$
$$T_2 = 0^\circ\,\mathrm{C}$$
$$\nu_1 = 440\,\mathrm{Hz}$$

we have that the speed of sound is proportional to the square root of the absolute temperature, however in the 0°C temperature region, the temperature dependance is given by:

$$c \approx c' + \left(\frac{c'}{2T'}\right) T_c,$$

where $T' = 273.2\,\mathrm{K}$ and T_c is the temperature measured in 0°C. From the text we see that $c'/2T' \approx 0.6\,\mathrm{m/sec{\cdot}{}^\circ C}$ and $c' = 331.3\,\mathrm{m/s}$. Thus

$$c(T_c) = (331.3\,\mathrm{m/s}) + (0.6\,\mathrm{m/sec} \cdot^\circ \mathrm{C})\,T_c,$$
$$c(25^\circ\mathrm{C}) = c_1 = (331.3\,\mathrm{m/s}) + (0.6\,\mathrm{m/sec} \cdot^\circ \mathrm{C})\,25^\circ\mathrm{C} = 346.3\,\mathrm{m/sec}, \text{ and}$$
$$c(0^\circ\mathrm{C}) = c_2 = (331.3\,\mathrm{m/s}) + (0.6\,\mathrm{m/sec} \cdot^\circ \mathrm{C})\,0^\circ\mathrm{C} = 331.3\,\mathrm{m/sec}.$$

As before, the physical dimensions of the pipe organ have not changed, and they determine the wavelength of the sound, so the frequency ν will be altered as c changes in the relation $\nu = c\lambda$. So the new frequency is given by:

$$\frac{\nu_2}{\nu_1} = \frac{c_2}{c_1} \quad \Longrightarrow \quad \nu_2 = \frac{c_2\nu_1}{c_1} == \frac{(331.3\,\mathrm{m/sec})(440\,\mathrm{Hz})}{(346.3\,\mathrm{m/sec})} = 420.9\,\mathrm{Hz}$$

Thus, at a temperature of 0°C, the pitch of the organ pipe will be about 420.9 Hz.

Alternatively, one could use either of the temperature/speed relations:

$$c = c'\left[1 + \frac{t_c}{T'}\right]^{\frac{1}{2}}$$
$$c^2 = \frac{\gamma R T_0}{M}$$

Headstart for next week, Week 03, starting Monday 2004/09/27:
– Read Chapter 2 “The Acoustic Plane Wave” in “Wave Phenomena” by Towne, omit 2-6
– – Section 2-4 “Simplified form of the equation for acoustic waves”
– – Section 2-5 “Detailed description of a progressive sinusoidal wave”
– Read Chapter 3 “Boundary value problems” in “Wave Phenomena” by Towne, omit 3-9
– – Section 3-1 “Reflection at a fixed end of transverse waves on a string”
– – Section 3-2 “Reflection of acoustic waves at a rigid surface”
– – Section 3-3 “Waves produced by the specified motion of a boundary surface”