Physics 202H - Introductory Quantum Physics I
Final Exam - Solutions

Fall 2004 **Name:** ____________________

Monday 2004/12/20 **Student Number:** ____________________

This examination paper includes 4 pages and 11 questions. You are responsible for ensuring that your copy of the paper is complete. Bring any discrepancy to the attention of your invigilator.

Special Instructions:

The only aids allowed are: a one (1) page double-sided hand-written formula sheet, and a calculator. When completed, turn in all exam booklets, the test paper, and the formula sheet.

Write your name and student number on the top of this paper AND on the front of your answer booklet AND on your formula sheet. Be prepared to present your student ID for verification.

Portable communications devices of all types (e.g. pagers, cellular phones, communicating calculators) are prohibited in the examination room. All such devices must be turned off prior to the start of the examination. A penalty of 5% of the exam mark may be assessed to anyone who fails to prevent a call from interrupting the examination.

Giving or receiving aid during an exam is a violation of university rules and may result in a failing grade and/or expulsion from the university.

Of possible use:

$$\int \sin ax\,\mathrm{d}x = -\frac{\cos ax}{a} \qquad \int \cos ax\,\mathrm{d}x = \frac{\sin ax}{a}$$

$$\int \sin^2 ax\,\mathrm{d}x = \frac{x}{2} - \frac{\sin 2ax}{4a} \qquad \int \cos^2 ax\,\mathrm{d}x = \frac{x}{2} + \frac{\sin 2ax}{4a}$$

$$\int x\sin ax\,\mathrm{d}x = \frac{\sin ax}{a^2} - \frac{x\cos ax}{a} \qquad \int x\cos ax\,\mathrm{d}x = \frac{\cos ax}{a^2} + \frac{x\sin ax}{a}$$

$$\int x^2\sin ax\,\mathrm{d}x = \frac{2x}{a^2}\sin ax + \left(\frac{2}{a^3} - \frac{x^2}{a}\right)\cos ax \qquad \int x^2\cos ax\,\mathrm{d}x = \frac{2x}{a^2}\sin ax + \left(\frac{2}{a^3} - \frac{x^2}{a}\right)\cos ax$$

$$\int x^2\cos^2 bx\,\mathrm{d}x = \frac{4b^3x^3 + 3(2b^2x^2 - 1)\sin 2bx + 6bx\cos 2bx}{24b^3}$$

$$\int_0^\infty \mathrm{e}^{-ax^2}\,\mathrm{d}x = \frac{1}{2}\sqrt{\frac{\pi}{a}}$$

$$x^3 - y^3 = (x-y)(x^2 + xy + y^2)$$

$$c = 2.99792458 \times 10^{8}\,\mathrm{m/s}$$

$$e = 1.602176462 \times 10^{-19}\,\mathrm{Coul}$$

$$h = 6.626068 \times 10^{-34}\,\mathrm{J\cdot s} = 4.1356668 \times 10^{-15}\,\mathrm{eV\cdot s}$$

$$\hbar = 1.05457148 \times 10^{-34}\,\mathrm{J\cdot s} = 6.58211814 \times 10^{-16}\,\mathrm{eV\cdot s}$$

$$m_e = 9.10938188 \times 10^{-31}\,\mathrm{kg} = 0.510998903\,\mathrm{MeV}/c^2$$

$$m_p = 1.67262158 \times 10^{-27}\,\mathrm{kg} = 938.271996\,\mathrm{MeV}/c^2$$

$$m_n = 1.6749286 \times 10^{-27}\,\mathrm{kg} = 939.565630\,\mathrm{MeV}/c^2$$

$$\sigma = 5.670400 \times 10^{-8}\,\mathrm{W/m^{2\circ}K^4}$$

$$\lambda_c = h/(m_e c) = 2.4263106 \times 10^{-12}\,\mathrm{m}$$

$$(4\pi\epsilon_0)^{-1} = 8.988 \times 10^{9}\,\mathrm{N\cdot m^2/Coul^2}$$

There is only this sentence on this page.

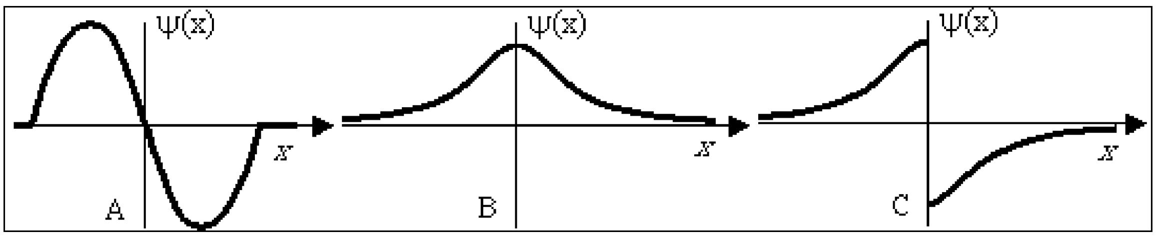
1. The wave function $\Psi(x,t) = A\sin(kx+\omega t)$ describes a particle moving in the positive x direction. [2]

   (a) True (b) **False**

2. An electron is confined to one-dimensional box of width L and is in its ground state. A proton is confined in another one-dimensional box also of width L also in its ground state. The wave functions have the same wavelength. [2]

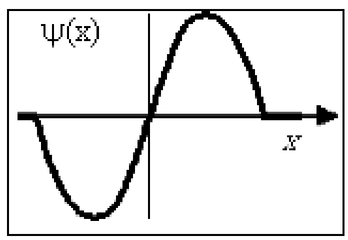
   (a) **True** (b) False

3. Which of the following is NOT a valid quantum mechanical wave function? [3]



   (a) A (b) B (c) **C**

4. Consider the $n = 2$ state for the particle confined to an infinite square well as shown.



   The particle is most likely to be found: [3]

   (a) in the right half of the box.
   (b) in the left half of the box.
   (c) **with equal probability in the left and right halves of the box.**

5. A certain blackbody radiates at a temperature of 5×10^4 °K. What area of radiating surface is required so that the radiated power is equal to 4×10^{12} W, approximately the power usage of humanity? [5]

   Solution: We need to find the radiated intensity, and use that to find the area:

$$\begin{aligned} R_T = \sigma T^4 &= \frac{P}{A} \\ A &= \frac{P}{\sigma T^4} \\ &= \frac{(4 \times 10^{12}\,\mathrm{W})}{(5.670400 \times 10^{-8}\,\mathrm{W/m^2{}^\circ K^4})(5 \times 10^4\,{}^\circ\mathrm{K})^4} \\ &= 11.286\ldots\ \mathrm{Wm}^2 \\ &\approx 11.3\,\mathrm{Wm}^2. \end{aligned}$$

   An area of about $11.3\,\mathrm{Wm}^2$ would be needed to radiate a power of 4×10^{12} W.

6. A region of space has a potential step such that particles have a wave function given by

$$\Psi(x,t) = \begin{cases} \dfrac{5a}{\sqrt{2}} \mathrm{e}^{\mathrm{i}(k_1 x - Et/\hbar)} + \dfrac{3a}{\sqrt{2}} \mathrm{e}^{\mathrm{i}(-k_1 x - Et/\hbar)}, & x < 0, \\ \\ \dfrac{8a}{\sqrt{2}} \mathrm{e}^{\mathrm{i}(k_2 x - Et/\hbar)}, & x > 0. \end{cases}$$

The incident particles, initially at $x \ll 0$, are initially travelling in the positive x direction.

(a) What fraction of the incident particles will be *reflected*? [5]

Solution: The given solution matches the format we have used in the course with wave amplitudes of A, B, and C. The reflection amplitude R will give us the fraction of incident particles reflected:

$$\begin{aligned} R &= \frac{v_1 B^* B}{v_1 A^* A} = \frac{B^* B}{A^* A} \\ &= \frac{\left(\frac{3a}{\sqrt{2}}\right)\left(\frac{3a}{\sqrt{2}}\right)}{\left(\frac{5a}{\sqrt{2}}\right)\left(\frac{5a}{\sqrt{2}}\right)} \\ &= \frac{(3)(3)}{(5)(5)} = \frac{9}{25} = 0.36. \end{aligned}$$

Of the incident particles, 36% or 9/25 of them will be reflected.

(b) What is k_2/k_1? [5]

Solution: Since 9/25 of the incident particles will be reflected, 16/25 or 64% will be transmitted. Using the relationship for the transmission coefficient T we have

$$\begin{aligned} T = 1 - R = \frac{v_2 C^* C}{v_1 A^* A} &= \frac{k_2 C^* C}{k_1 A^* A} \\ 1 - \frac{9}{25} &= \frac{k_2 \left(\frac{8a}{\sqrt{2}}\right)\left(\frac{8a}{\sqrt{2}}\right)}{k_1 \left(\frac{5a}{\sqrt{2}}\right)\left(\frac{5a}{\sqrt{2}}\right)} \\ \frac{16}{25} &= \frac{k_2 (8)(8)}{k_1 (5)(5)} = \frac{k_2 64}{k_1 25} \\ 16 &= 64 \frac{k_2}{k_1} \\ \frac{k_2}{k_1} &= \frac{1}{4}. \end{aligned}$$

Alternatively, one could use the continuity of the first derivative of the eigenfunction at $x = 0$ to get

$$\begin{aligned} \left[\frac{\mathrm{d}\psi_1(x)}{\mathrm{d}x}\right]_{x=0} &= \left[\frac{\mathrm{d}\psi_2(x)}{\mathrm{d}x}\right]_{x=0} \\ k_1 \frac{5a}{\sqrt{2}} - k_1 \frac{3a}{\sqrt{2}} &= k_2 \frac{8a}{\sqrt{2}} \\ 2k_1 &= 8k_2 \\ \frac{k_2}{k_1} &= \frac{1}{4}. \end{aligned}$$

7. In a photoelectric effect experiment on a certain metal, it is observed that incident light of wavelength 413 nm causes electrons to be ejected from the metal's surface with a maximum kinetic energy of 3.2×10^{-19} J. What is the longest wavelength of light that will eject electrons from this metal? [5]

   Solution: The photon has energy of $h\nu = hc/\lambda$, so the kinetic energy is related to the work function of the material by

$$
\begin{aligned}
K &= h\nu - w_0 \\
&= \frac{hc}{\lambda} - w_0 \\
w_0 &= \frac{hc}{\lambda} - K \\
&= \frac{(6.626068 \times 10^{-34}\,\mathrm{J \cdot s})(2.99792458 \times 10^{8}\,\mathrm{m/s})}{(413\,\mathrm{nm})} - (3.2 \times 10^{-19}\,\mathrm{J}) \\
&= (4.8098 \times 10^{-19}\,\mathrm{J}) - (3.2 \times 10^{-19}\,\mathrm{J}) \\
w_0 &= 1.6098 \times 10^{-19}\,\mathrm{J}
\end{aligned}
$$

   The longest wavelength will occur when the kenetic energy of the emitted electron is a minimum, zero.

$$
\begin{aligned}
0 &= h\nu_{min} - w_0 \\
&= \frac{hc}{\lambda_{max}} - w_0 \\
w_0 &= \frac{hc}{\lambda_{max}} \\
\lambda_{max} &= \frac{hc}{w_0} \\
&= \frac{(6.626068 \times 10^{-34}\,\mathrm{J \cdot s})(2.99792458 \times 10^{8}\,\mathrm{m/s})}{(1.6098 \times 10^{-19}\,\mathrm{J})} \\
&= 1.23397 \times 10^{-6}\,\mathrm{m} \approx 1.23\,\mu\mathrm{m} = 1230\,\mathrm{nm}.
\end{aligned}
$$

   The longest wavelength of light that will eject electrons from this metal is about 1230 nm

8. A particle is in an region of space where it has a wave function given by

$$
\Psi(x,t) = \begin{cases} 0, & x < 0, \\ A\mathrm{e}^{-\kappa x}\mathrm{e}^{\mathrm{i}Et/\hbar}, & x > 0. \end{cases}
$$

   What is the value of A? [5]

   Solution: We need to normalize the wave function by setting the integral over all space of $P(x,t)\mathrm{d}x = \Psi^*(x,t)\Psi(x,t)\mathrm{d}x$ to 1.

$$
\begin{aligned}
1 &= \int_{-\infty}^{+\infty} P(x,t)\mathrm{d}x = \int_{-\infty}^{+\infty} \Psi^*(x,t)\Psi(x,t)\mathrm{d}x \\
&= \int_{0}^{+\infty} \left(A\mathrm{e}^{-\kappa x}\mathrm{e}^{-\mathrm{i}Et/\hbar}\right)\left(A\mathrm{e}^{-\kappa x}\mathrm{e}^{\mathrm{i}Et/\hbar}\right)\mathrm{d}x = A^2 \int_{0}^{+\infty} \mathrm{e}^{-2\kappa x}\mathrm{d}x \\
&= A^2 \left[\frac{\mathrm{e}^{-2\kappa x}}{-2\kappa}\right]_0^{+\infty} = A^2 \left[\frac{\mathrm{e}^{-2\kappa(\infty)}}{-2\kappa} - \frac{\mathrm{e}^{-2\kappa(x)}}{-2\kappa}\right] = A^2\left[0 + \frac{1}{2\kappa}\right] \\
1 &= \frac{A^2}{2\kappa} \quad \Longrightarrow \quad A = \sqrt{2\kappa}
\end{aligned}
$$

   For $\Psi(x,t)$ to be properly normalized, A must have a value of $\sqrt{2\kappa}$.

9. A particle of mass m is confined to a harmonic oscillator potential given by $V = mx^2\omega^2/2$, where $\omega^2 = K/m$ and K is the force constant. The particle is in a state described by the wave function

$$\Psi(x,t) = A\mathrm{e}^{\left(\frac{-mx^2\omega}{2\hbar} - \mathrm{i}\frac{\omega t}{2}\right)}.$$

Verify that this is a solution of Schroedinger's equation. [5]

Solution: Taking the given function, the various derivatives of $\Psi(x,t)$ are:

$$\frac{\partial \Psi(x,t)}{\partial t} = \left(-\mathrm{i}\frac{\omega}{2}\right) A\mathrm{e}^{\left(\frac{-mx^2\omega}{2\hbar} - \mathrm{i}\frac{\omega t}{2}\right)} = \left(-\mathrm{i}\frac{\omega}{2}\right)\Psi \tag{9.1}$$

$$\begin{aligned}
\frac{\partial \Psi(x,t)}{\partial x} &= \left(\frac{-2mx\omega}{2\hbar}\right) A\mathrm{e}^{\left(\frac{-mx^2\omega}{2\hbar} - \mathrm{i}\frac{\omega t}{2}\right)} \\
\therefore \Psi' &= \left(\frac{-mx\omega}{\hbar}\right)\Psi \\
\frac{\partial^2 \Psi(x,t)}{\partial x^2} &= \left(\frac{-mx\omega}{\hbar}\right)'\Psi + \left(\frac{-mx\omega}{\hbar}\right)\Psi' \\
&= \left(\frac{-m\omega}{\hbar}\right)\Psi + \left(\frac{-mx\omega}{\hbar}\right)\left(\frac{-mx\omega}{\hbar}\right)\Psi \\
&= \left(\frac{-m\omega}{\hbar}\right)\Psi + \left(\frac{mx\omega}{\hbar}\right)^2\Psi
\end{aligned} \tag{9.2}$$

Building on (9.2) and using the given potential we get:

$$\begin{aligned}
-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi &= \frac{-\hbar^2}{2m}\left[\left(\frac{-m\omega}{\hbar}\right)\Psi + \left(\frac{mx\omega}{\hbar}\right)^2\Psi\right] + \frac{mx^2\omega^2}{2}\Psi \\
&= \left[\frac{\hbar\omega}{2} - \frac{mx^2\omega}{2} + \frac{mx^2\omega^2}{2}\right]\Psi = \frac{\hbar\omega}{2}\Psi
\end{aligned} \tag{9.3}$$

Taking (9.1) and multiplying by $\mathrm{i}\hbar$ we get

$$\mathrm{i}\hbar\frac{\partial\Psi}{\partial t} = (\mathrm{i}\hbar)\left(-\mathrm{i}\frac{\omega}{2}\right)\Psi = \frac{\hbar\omega}{2}\Psi \tag{9.4}$$

Since (9.3) and (9.4) are equal, we can put them together to get

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi = \mathrm{i}\hbar\frac{\partial\Psi}{\partial t},$$

which is the Schroedinger's equation,

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi(x,t)}{\partial x^2} + V(x,t)\Psi(x,t) = \mathrm{i}\hbar\frac{\partial\Psi(x,t)}{\partial t}.$$

Thus the given function $\Psi(x,t)$ is a solution to the Schroedinger's equation for the given potential $V(x,t)$.

10. A particle moves in an infinite potential well described by

$$V(x) = \begin{cases} 0, & |x| > a/2, \\ \infty, & |x| \leq a/2. \end{cases}$$

The eigenfunctions are of the form $\psi_n(x) = A_n \cos(k_n x)$, or $\psi_n(x) = B_n \sin(k_n x)$, depending on the value of n. For $n = 3$, $\psi_3(x) = (\sqrt{2/a}) \cos(3\pi x/a)$ for $|x| \leq a/2$ and $\psi_3(x) = 0$ for $|x| > a/2$.

(a) What are the expectation values of x and x^2 in the $n = 3$ state. [5]

Solution: The expectation value of x is calculated by integrating $\psi_3^* x \psi_3 \mathrm{d}x$ over all space, but since the wave function is zero outside of $\pm a/2$ we only need to integrate in this region.

$$\begin{aligned}
\overline{x} &= \int_{-a/2}^{+a/2} \psi^*(x) x \psi(x) \mathrm{d}x \\
&= \int_{-a/2}^{+a/2} \sqrt{\frac{2}{a}} \cos\left(\frac{3\pi x}{a}\right) x \sqrt{\frac{2}{a}} \cos\left(\frac{3\pi x}{a}\right) \mathrm{d}x \\
&= \frac{2}{a} \int_{-a/2}^{+a/2} x \cos^2\left(\frac{3\pi x}{a}\right) \mathrm{d}x \qquad u = \frac{3\pi x}{a}, \mathrm{d}u = \frac{3\pi}{a} \mathrm{d}x \\
&= \frac{2a}{9\pi^2} \int_{-3\pi/2}^{+3\pi/2} \underbrace{\overbrace{u}^{\text{odd}} \overbrace{\cos^2 u}^{\text{even}}}_{\text{odd}} \, \mathrm{d}u = 0 \\
\overline{x} &= 0.
\end{aligned}$$

The expectation value of x^2 is calculated by integrating $\psi_3^* x^2 \psi_3 \mathrm{d}x$ over all space, but since the wave function is zero outside of $\pm a/2$ we only need to integrate in this region.

$$\begin{aligned}
\overline{x^2} &= \int_{-a/2}^{+a/2} \psi^*(x) x^2 \psi(x) \mathrm{d}x \\
&= \int_{-a/2}^{+a/2} \sqrt{\frac{2}{a}} \cos\left(\frac{3\pi x}{a}\right) x^2 \sqrt{\frac{2}{a}} \cos\left(\frac{3\pi x}{a}\right) \mathrm{d}x \\
&= \frac{2}{a} \int_{-a/2}^{+a/2} x^2 \cos^2\left(\frac{3\pi x}{a}\right) \mathrm{d}x \qquad u = \frac{3\pi x}{a}, \mathrm{d}u = \frac{3\pi}{a} \mathrm{d}x \\
&= \frac{2a^2}{27\pi^3} \int_{-3\pi/2}^{+3\pi/2} \underbrace{u^2 \cos^2 u}_{\text{even}} \, \mathrm{d}u = \frac{4a^2}{27\pi^3} \int_0^{+3\pi/2} u^2 \cos^2 u \, \mathrm{d}u \\
&= \frac{4a^2}{27\pi^3} \left[\frac{4b^3x^3 + 3(2b^2x^2 - 1) \sin 2bx + 6bx \cos 2bx}{24b^3} \right]_0^{+3\pi/2} \qquad b = 1 \\
&= \frac{a^2}{(27)(6)\pi^3} \left[4x^3 + 3(2x^2 - 1) \sin 2x + 6x \cos 2x \right]_0^{+3\pi/2} \\
&= \frac{a^2}{(27)(6)\pi^3} \left[4\left(\frac{3\pi}{2}\right)^3 + 3(2\left(\frac{3\pi}{2}\right)^2 - 1) \sin 2\left(\frac{3\pi}{2}\right) + 6\left(\frac{3\pi}{2}\right) \cos 2\left(\frac{3\pi}{2}\right) \right] \\
&= \frac{a^2}{(27)(6)\pi^3} \left[4\left(\frac{3\pi}{2}\right)^3 + 0 - 6\left(\frac{3\pi}{2}\right) \right] = \frac{a^2}{(27)(6)\pi^3} \left[\frac{27\pi^3}{2} - 9\pi \right] \\
&= \frac{a^2}{6} \left(\frac{1}{2} - \frac{1}{3\pi^2} \right) = \frac{a^2}{12} - \frac{a^2}{18\pi^2} \approx (0.0777)a^2
\end{aligned}$$

(b) What are the expectation values of p and p^2 in the $n = 3$ state. [5]

Solution: To calculate the expectation value for momentum, we need to use the momentum operator $-\mathrm{i}\hbar(\partial/\partial x)$.

$$\begin{aligned}
\overline{p} &= \int_{-a/2}^{+a/2} \psi^*(x) \left(-\mathrm{i}\hbar\frac{\partial}{\partial x}\right) \psi(x)\mathrm{d}x \\
&= \int_{-a/2}^{+a/2} \sqrt{\frac{2}{a}} \cos\left(\frac{3\pi x}{a}\right) \left(-\mathrm{i}\hbar\frac{\partial}{\partial x}\right) \sqrt{\frac{2}{a}} \cos\left(\frac{3\pi x}{a}\right)\mathrm{d}x \\
&= \frac{2}{a}\int_{-a/2}^{+a/2} \cos\left(\frac{3\pi x}{a}\right) \frac{\mathrm{i}\hbar 3\pi}{a} \sin\left(\frac{3\pi x}{a}\right)\mathrm{d}x \qquad u = \frac{3\pi x}{a}, \mathrm{d}u = \frac{3\pi}{a}\mathrm{d}x \\
&= \frac{-\mathrm{i}\hbar 2}{a}\int_{-3\pi/2}^{+3\pi/2} \underbrace{\overbrace{\cos u}^{\text{even}}\,\overbrace{\sin u}^{\text{odd}}}_{\text{odd}}\,\mathrm{d}u = 0 \\
\overline{p} &= 0.
\end{aligned}$$

To calculate the expectation value for p^2 we need to use $-\hbar^2(\partial^2/\partial x^2)$.

$$\begin{aligned}
\overline{p^2} &= \int_{-a/2}^{+a/2} \psi^*(x) \left(-\hbar^2\frac{\partial^2}{\partial x^2}\right) \psi(x)\mathrm{d}x \\
&= \int_{-a/2}^{+a/2} \sqrt{\frac{2}{a}} \cos\left(\frac{3\pi x}{a}\right) \left(-\hbar^2\frac{\partial^2}{\partial x^2}\right) \sqrt{\frac{2}{a}} \cos\left(\frac{3\pi x}{a}\right)\mathrm{d}x \\
&= \frac{2}{a}\int_{-a/2}^{+a/2} -\hbar^2 \left(\frac{-9\pi^2}{a^2}\right) \cos^2\left(\frac{3\pi x}{a}\right)\mathrm{d}x \\
&= \frac{18\pi^2\hbar^2}{a^3}\int_{-a/2}^{+a/2} \cos^2\left(\frac{3\pi x}{a}\right)\mathrm{d}x \qquad u = \frac{3\pi x}{a}, \mathrm{d}u = \frac{3\pi}{a}\mathrm{d}x \\
&= \frac{6\pi\hbar^2}{a^2}\int_{-3\pi/2}^{+3\pi/2} \underbrace{\cos^2 u}_{\text{even}}\,\mathrm{d}u = \frac{12\pi\hbar^2}{a^2}\int_{0}^{+3\pi/2} \cos^2 u\,\mathrm{d}u \\
&= \frac{12\pi\hbar^2}{a^2}\left[\frac{u}{2} + \frac{\sin(2u)}{4}\right]_0^{+3\pi/2} = \frac{12\pi\hbar^2}{a^2}\left[\frac{3\pi}{4} + 0 - 0 - 0\right] \\
\overline{p^2} &= \frac{9\pi^2\hbar^2}{a^2} = \left(\frac{3\pi\hbar}{a}\right)^2 = \left(\frac{3h}{2a}\right)^2
\end{aligned}$$

11. Do **ONE** of the part (a) or part (b). [15]

(a) In a Compton scattering event, the *scattered* photon was found to have an energy of 120 keV, and the recoil electron was given a kinetic energy of 40 keV.

i. What was the wavelength of the incident photon? [5]

Solution: The energy of the incident photon is $E_{ip} = E_{sp} + K_{re} = 120\,\text{keV} + 40\,\text{keV} = 160\,\text{keV}$. This energy is related to the wavelength of the photon by

$$\begin{aligned}
E_{ip} &= h\nu_{ip} = \frac{hc}{\lambda_{ip}} \\
\lambda_{ip} &= \frac{hc}{E_{ip}} = \frac{(4.1356668 \times 10^{-15}\,\text{eV}\cdot\text{s})(2.99792458 \times 10^{8}\,\text{m/s})}{1.6 \times 10^{5}\,\text{eV}} \\
&= 7.7490107 \times 10^{-12}\,\text{m} \approx 7.75 \times 10^{-12}\,\text{m}.
\end{aligned}$$

The wavelength of the incident photon is about 7.75 pm.

ii. What was the scattering angle θ for the 120 keV photon? [5]

Solution: The wavelength of the scattered photon gives us $\Delta\lambda$ by

$$\begin{aligned}
\lambda_{sp} &= \frac{hc}{E_{sp}} = \frac{(4.1356668 \times 10^{-15}\,\text{eV}\cdot\text{s})(2.99792458 \times 10^{8}\,\text{m/s})}{1.2 \times 10^{5}\,\text{eV}} \\
&= 0.103320 \times 10^{-12}\,\text{m} \\
\Delta\lambda &= \lambda_c(1-\cos\theta) = \lambda_{sp} - \lambda_{ip} = 2.58300 \times 10^{-12}\,\text{m} \\
\frac{\Delta\lambda}{\lambda_c} &= \frac{2.58300}{2.4263106} = 1.06458 \\
1-\cos\theta &= 1.06458 \\
\cos\theta &= -0.0645811042 \\
\theta &= \cos^{-1}(-0.0645811042) = 1.635422407\,\text{rad} \approx 93.7^\circ.
\end{aligned}$$

The 120 keV photon is scattered at an angle of about 93.7°. Note that this is greater than 90°, indicating that the scattered photon is coming slightly back in the same direction as the incident photon.

Alternatively, one can calculate $\Delta\lambda/\lambda_c$ via

$$\begin{aligned}
\Delta\lambda &= \lambda_{sp} - \lambda_{ip} = hc\left(\frac{1}{E_{sp}} - \frac{1}{E_{ip}}\right) \\
\Delta\lambda &= \lambda_c(1-\cos\theta) = \frac{h}{m_e c}(1-\cos\theta) \\
1-\cos\theta &= \frac{\Delta\lambda}{\lambda_c} = \Delta\lambda\frac{m_e c}{h} \\
&= hc\left(\frac{1}{E_{sp}} - \frac{1}{E_{ip}}\right)\frac{m_e c}{h} = m_e c^2\left(\frac{1}{E_{sp}} - \frac{1}{E_{ip}}\right) \\
&= (510.998903\,\text{keV})\left(\frac{1}{120\,\text{keV}} - \frac{1}{160\,\text{keV}}\right) \\
1-\cos\theta &= 1.06458.
\end{aligned}$$

iii. What angle ϕ does the path of the scattered electron make with the direction of the incident photon? [5]

Solution: The kinetic energy of the recoil electron is less than 10% of its rest mass energy, so non-relativistic relations will be fairly accurate. The momentum of the electron in the y direction, perpendicular to the initial photon's direction of travel, must be equal and opposite to the momentum of the scattered photon in the y direction, since there is zero total momentum in the y direction. Thus

$$\begin{aligned}
p_{ipy} = 0 &= p_{rey} + p_{spy} \\
p_{rey} &= -p_{spy} \\
p_{re}\sin\phi &= -p_{sp}\sin\theta \\
\sqrt{2m_e E_{re}}\sin\phi &= -\frac{E_{sp}}{c}\sin\theta \\
\sin\phi &= -\frac{E_{sp}}{\sqrt{2m_e c^2 E_{re}}}\sin\theta \\
&= -\frac{(120\,\mathrm{keV})}{\sqrt{2(510.998903\,\mathrm{keV})(40\,\mathrm{keV})}}\sin(1.635422407) \\
&= -0.5922686084 \\
\phi &= \sin^{-1}(-0.5922686084) = -0.6338714937\,\mathrm{rad} \approx -36.3^\circ.
\end{aligned}$$

The electron is scattered about 36.3°from the path of the incident photon, with the negative sign indicating that the electron is scattered on the one side of the x axis as defined by the direction of the incident photon, while the scattered photon is on the opposite side.

(b) Electrons are accelerated through an electric potential V and then fall on a pair of slits that have a separation of 100 nm. The resultant interference pattern indicates that the electrons have a wavelength of 1.0 nm.

i. What is the momentum of one of the electrons? [5]

Solution: The momentum is related to the de Broglie wavelength by

$$p = \frac{h}{\lambda} = \frac{(6.626068 \times 10^{-34}\,\mathrm{J\cdot s})}{(1.0 \times 10^{-9}\,\mathrm{m})} = (6.626068\times 10^{-25}\,\mathrm{kg\cdot m/s}) \approx 6.63\times 10^{-25}\,\mathrm{kg\cdot m/s}.$$

The momentum of one of the electrons is about $6.63 \times 10^{-25}\,\mathrm{kg\cdot m/s}$.

ii. What is the accelerating electric potential V? [5]

Solution: With a momentum of about $6.63 \times 10^{-25}\,\mathrm{kg\cdot m/s}$ and a mass of $m_e = 9.10938188 \times 10^{-31}\,\mathrm{kg}$, the electron's velocity of $v = p/m = 7.274 \times 10^5\,\mathrm{m/s} \ll \mathrm{c}$, thus we can use non-relativistic relations. The kinetic energy gives us

$$\begin{aligned}
K = eV &= \frac{p^2}{2m_e}. \\
V &= \frac{p^2}{2em_e} = \frac{(6.626068 \times 10^{-25}\,\mathrm{kg\cdot m/s})^2}{2(1.602176462 \times 10^{-19}\,\mathrm{Coul})(9.10938188 \times 10^{-31}\,\mathrm{kg})} \\
&= 1.5041201366\,\mathrm{V} \approx 1.50\,\mathrm{V}.
\end{aligned}$$

The accelerating electric potential is about 1.50 V.

iii. Using the uncertainty principle for the electrons after passing through the slits, what is the minimum spread in the electron's momentum in the direction parallel to the plane of the slits and perpendicular to the average path of the electron? [5]

Solution: Since the slits are separated by 100 nm, the uncertainty in the position in the direction parallel to the plane of the slits is $\Delta x = 100\,\mathrm{nm}$. Thus

$$\begin{aligned}\Delta x \Delta p_x &\geq \frac{\hbar}{2}\\ \Delta p_x &\geq \frac{\hbar}{2\Delta x} = \frac{(1.05457148 \times 10^{-34}\,\mathrm{J \cdot s})}{2(1.0 \times 10^{-7}\,\mathrm{m})}\\ &\geq 5.2728574 \times 10^{-28}\,\mathrm{kg \cdot m/s} \approx 5.27 \times 10^{-28}\,\mathrm{kg \cdot m/s}.\end{aligned}$$

The minimum spread in the electron's momentum in the direction parallel to the plane of the slits is about $5.27 \times 10^{-28}\,\mathrm{kg \cdot m/s}$.