

Physics 202H - Introductory Quantum Physics I

Homework #10 - Solutions

Fall 2004

Due 5:01 PM, Monday 2004/11/29

[40 points total]

“Journal” questions. Briefly share your thoughts on the following questions:

- How do you feel about the usefulness and/or effectiveness of these “Journal” types of activities? What do you think their best aspect has been? What change to their format or content might improve their usefulness and/or effectiveness? Why?
 - Any comments about this week’s activities? Course content? Assignment? Lab?
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1. For a free particle the wave function is $\Psi(x, t) = Ae^{i(kx - \omega t)}$. What value do you get for A if you normalize this wave function over all space? How can we address this difficulty? Limit your discussion to about 50 words or so. [10]

Solution: In order to find A we need to integrate $\Psi^*\Psi dx$ over all space. This integral should have a value of one since we have exactly 100% probability of finding the particle *somewhere*. However for the free particle wave function, $\Psi^*\Psi = A^2$ so the integral over all space will be infinite for any non-zero value of A . There is no value for A which will normalize the wave function, although setting $A = 0$ would save us from the infinity it would not give us a normalized wave function. To address this difficulty, we need to recognize that the given wave function specifies a particle with definite momentum, but no defined location. Since any wave function that we are interested in will have a finite spacial extent we will need to limit the non-zero magnitude values of Ψ so some reasonable finite spacial extent. For any finite volume of space, the integral of $\Psi^*\Psi dx$ will not be finite, so we will always be able to find a non-zero value for A in order to properly normalize the wave function. This method of normalization is often called “box normalization” since the wave function is set to zero for all spacial values outside of some arbitrary “box.” In actual practice, we rarely have to actually normalize the wave function, since in most cases we are interested in quantities that are independent of the specific value of A as long as A is non-zero and finite.

2. (From Eisberg & Resnick, Q 6-11, pg 227) Exactly what is meant by the statement that the reflection coefficient is one for a particle incident on a potential step with total energy less than the step height? What is meant by the statement that the reflection coefficient is less than one but greater than zero if the total energy is greater than the step height? Can the reflection coefficient ever be greater than one? Limit your discussion to about 50 words or so. [10]

Solution: The reflection coefficient is a measurement of the probability of an incident particle being reflected (or alternatively a measurement of the fraction of incident particles that are reflected). Thus for the case when $E < V_0$ thus $R = 1$, we see that *all* incident particles will be reflected, there is zero chance that any particles will be transmitted. This is consistent with the classically expected results - when particles do not have enough energy to get past a barrier, they bounce back.

For the case when $E > V_0$ the classically expected result would be that *no* particles are reflected, but the quantum result is that there is a non-zero chance that a particle will be reflected. Just like for water/string/EM waves, a change in the wave medium results in partial reflection and partial transmission - so in the quantum wave case we get a reflection coefficient that is non-zero even if the particle has more energy than the step height.

Since the reflection coefficient is a measure of probability or fraction of incoming particles, we can never have it be greater than one since particles are not being created or destroyed we must have that the sum of the reflection and the transmission coefficient are one. (Actually, for certain formalisms R and T can be complex numbers, so they could be positive or negative in certain situations. Thus we should really say that the sum of the *magnitudes* of R and T must be one.)

3. (From Eisberg & Resnick, P 6-2, pg 228) Repeat the step potential calculation of Section 6-4, but with the particle initially in the region $x > 0$ where $V(x) = V_0$, and traveling in the direction of decreasing x towards the point $x = 0$ where the potential steps down to its value $V(x) = 0$ in the region $x < 0$. Show that the transmission and reflection coefficients are the same as those obtained in Section 6-4. [10]

Solution: The potential energy function $V(x)$ with $E > V_0$ looks like figure 1.

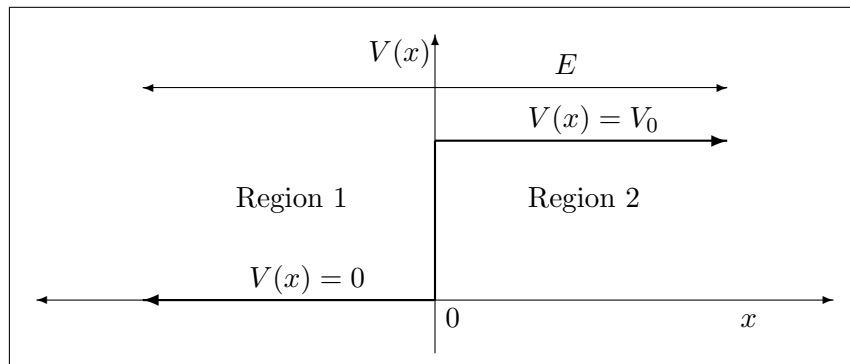


Figure 1: Step potential, $E > V_0$

Following Eisberg & Resnick, the time-independant Schroedinger equation has different forms for each region, specifically

$$\begin{aligned} -\frac{\hbar^2}{2m} \frac{d^2\psi_1(x)}{dx^2} &= E\psi_1(x) & x < 0, & \text{ Region 1} \\ -\frac{\hbar^2}{2m} \frac{d^2\psi_2(x)}{dx^2} &= (E - V_0)\psi_2(x) & x > 0, & \text{ Region 2.} \end{aligned}$$

Solutions to the TISE in each region are, most generally

$$\begin{aligned} \psi_1(x) &= Ae^{ik_1x} + Be^{-ik_1x} & x < 0, & \text{ Region 1} \\ \psi_2(x) &= Ce^{ik_2x} + De^{-ik_2x} & x > 0, & \text{ Region 2,} \end{aligned}$$

where

$$k_1 = \frac{\sqrt{2mE}}{\hbar} \quad \text{and} \quad k_2 = \frac{\sqrt{2m(E - v_0)}}{\hbar}.$$

The derivatives of the functions are

$$\begin{aligned} \frac{d\psi_1(x)}{dx} &= ik_1Ae^{ik_1x} - ik_1Be^{-ik_1x} & x < 0, & \text{ Region 1} \\ \frac{d\psi_2(x)}{dx} &= ik_2Ce^{ik_2x} - ik_2De^{-ik_2x} & x > 0, & \text{ Region 2,} \end{aligned}$$

Since ψ and its derivative are finite, single-valued, and continuous, we can apply the following continuity relationships

$$[\psi_1(x)]_{x=0} = [\psi_2(x)]_{x=0} \quad \text{and} \quad \left[\frac{d\psi_1(x)}{dx} \right]_{x=0} = \left[\frac{d\psi_2(x)}{dx} \right]_{x=0}. \quad (3.1)$$

Applying the continuity relationships (3.1) gives us

$$\begin{aligned} A + B &= C + D \\ ik_1A - ik_1B &= ik_2C - ik_2D. \end{aligned}$$

We also note that since the particles are coming in from the left, and the term Ae^{ik_1x} represents particles moving in from the right, the constant A must be zero, while D is not zero. This is opposite from the example in Eisberg & Resnick with the incident particles from the right. Now we have the relationships are now

$$\begin{aligned} B &= C + D & (3.2) \\ -k_1B &= k_2(C - D). & (3.3) \end{aligned}$$

Solving for C in terms of D can be done by placing (3.2) into (3.3) to get

$$\begin{aligned} -k_1(C + D) &= k_2(C - D) \\ -k_1C - k_1D &= k_2C - k_2D \\ k_2C + k_1C &= k_2D - k_1D \\ C(k_2 + k_1) &= D(k_2 - k_1) \\ C &= D \frac{k_2 - k_1}{k_2 + k_1}. \end{aligned} \quad (3.4)$$

To solve for B in terms of D we can put (3.4) into (3.2) to give us

$$\begin{aligned} B &= D \frac{k_2 - k_1}{k_2 + k_1} + D \\ &= D \frac{k_2 - k_1}{k_2 + k_1} + D \frac{k_2 + k_1}{k_2 + k_1} \\ &= D \frac{2k_2}{k_2 + k_1}. \end{aligned} \quad (3.5)$$

This gives us enough to find the reflection and transmission coefficients. Since we know that $v = p/m = \hbar k/m$ we have:

$$\begin{aligned} R &= \frac{v_2 C^* C}{v_2 D^* D} & T &= \frac{v_1 B^* B}{v_2 D^* D} \\ &= \frac{C^* C}{D^* D} & &= \frac{k_1 B^* B}{k_2 D^* D} \\ &= \frac{(k_2 - k_1)^2}{(k_2 + k_1)^2} & &= \frac{k_1 (2k_2)^2}{k_2 (k_2 + k_1)^2} \\ R &= \frac{(k_2 - k_1)^2}{(k_2 + k_1)^2} = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2} & T &= \frac{4k_2 k_1}{(k_2 + k_1)^2} = \frac{4k_1 k_2}{(k_1 + k_2)^2}. \end{aligned}$$

These are exactly the same as in Eisberg & Resnick with the incident particles from the right.

Alternatively, one could have done the exact manipulations that Eisberg & Resnick did for the incident particles from the left (keeping A , B , and C as they did) but swapped the values of k_1 and k_2 to indicate that in region 2 the potential energy was $V(x) = 0$ and in region 1 the potential energy was $V(x) = V_0$. This would again have resulted in the above values for R and T .

4. (From Eisberg & Resnick, P 6-3, pg 228) Prove that the sum of the reflection and transmission coefficients equals one (Eisberg & Resnick, Equation 6-43, pg 197), for the case of a step potential with $E > V_0$. [10]

Solution: For the case of a step potential with $E > V_0$, we have expressions for both R and T , namely

$$R = \frac{v_1 B^* B}{v_1 A^* A} = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2} \qquad T = \frac{v_2 C^* C}{v_1 A^* A} = \frac{4k_1 k_2}{(k_1 + k_2)^2}.$$

If we add them together we get

$$\begin{aligned} R + T &= \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2} + \frac{4k_1 k_2}{(k_1 + k_2)^2} = \frac{(k_1 - k_2)^2 + 4k_1 k_2}{(k_1 + k_2)^2} \\ &= \frac{(k_1^2 - 2k_1 k_2 + k_2^2) + 4k_1 k_2}{k_1^2 + 2k_1 k_2 + k_2^2} = \frac{k_1^2 + 2k_1 k_2 + k_2^2}{k_1^2 + 2k_1 k_2 + k_2^2} \\ \therefore R + T &= 1. \end{aligned}$$

Headstart for next week, Week 11, starting Monday 2004/11/29:

- Read Chapter 6 "Solutions of Time-Independent Schroedinger Equation" in Eisberg & Resnick
- - Section 6.5 "The Barrier Potential"
- - Section 6.6 "Examples of Barrier Penetration by Particles"
- - Section 6.7 "The Square Well Potential"
- - Section 6.8 "The Infinite Square Well Potential"