

Physics 202H - Introductory Quantum Physics I

Homework #09 - Solutions

Fall 2004

Due 5:01 PM, Monday 2004/11/22

[50 points total]

“Journal” questions. Briefly share your thoughts on the following questions:

- Within the subject matter of this course, what do you think the best methods of evaluating student knowledge and/or skills would be? What single change to how we do evaluation in this course do you think would be best? What is the best feature of the evaluation method used in this course? Why?
 - Any comments about this week’s activities? Course content? Assignment? Lab?
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1. (From Eisberg & Resnick, Q 5-25, pg 168) Explain briefly the meaning of a well-behaved eigenfunction in the context of Schroedinger quantum mechanics. Why do we need the eigenfunction to be well-behaved? Limit your discussion to about 50 words or so. [10]

Solution: The eigenfunction of the time-independent Schroedinger equation is the spatial part of the wave function which is a solution to the Schroedinger equation for a time-independent potential energy function. Since the wave function is used to calculate actual quantities that can be measured in the lab, it is reasonable to insist that the calculated quantities are “well-behaved”, namely that they are not discontinuous, multi-valued, or infinite. Thus it is necessary that the eigenfunctions (and their first derivatives) are also “well-behaved”, namely that they are continuous, single-valued, and finite in order that measurable quantities which can be evaluated from the eigenfunctions will also be well-behaved.

2. (From Eisberg & Resnick, P 5-1, pg 169) If the wave functions $\Psi_1(x, t)$, $\Psi_2(x, t)$, and $\Psi_3(x, t)$ are three solutions to the Schroedinger equation for a particular potential $V(x, t)$, show that the arbitrary linear combination $\Psi(x, t) = c_1\Psi_1(x, t) + c_2\Psi_2(x, t) + c_3\Psi_3(x, t)$ is also a solution to that equation. [10]

Solution: Since the three wave functions are solutions to the Schroedinger equation, we know that

$$\begin{aligned} -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi_1(x, t)}{\partial x^2} + V(x, t)\Psi_1(x, t) &= i\hbar \frac{\partial \Psi_1(x, t)}{\partial t} \\ -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi_1(x, t)}{\partial x^2} + V(x, t)\Psi_1(x, t) - i\hbar \frac{\partial \Psi_1(x, t)}{\partial t} &= 0 \end{aligned} \tag{2.1}$$

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \Psi_2(x, t)}{\partial x^2} + V(x, t)\Psi_2(x, t) - i\hbar \frac{\partial \Psi_2(x, t)}{\partial t} = 0 \tag{2.2}$$

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \Psi_3(x, t)}{\partial x^2} + V(x, t)\Psi_3(x, t) - i\hbar \frac{\partial \Psi_3(x, t)}{\partial t} = 0. \tag{2.3}$$

If we try the wave function $\Psi(x, t)$ in the Schroedinger equation we get:

$$-\frac{\hbar^2}{2m} \frac{\partial^2 [c_1\Psi_1 + c_2\Psi_2 + c_3\Psi_3]}{\partial x^2} + V(x, t) [c_1\Psi_1 + c_2\Psi_2 + c_3\Psi_3] = i\hbar \frac{\partial [c_1\Psi_1 + c_2\Psi_2 + c_3\Psi_3]}{\partial t}$$

The constants do not effect the derivatives, and so we can bring them outside after gathering like terms together and we get

$$\begin{aligned} c_1 \left[-\frac{\hbar^2}{2m} \frac{\partial^2 \Psi_1}{\partial x^2} + V(x, t)\Psi_1 - i\hbar \frac{\partial \Psi_1}{\partial t} \right] &+ c_2 \left[-\frac{\hbar^2}{2m} \frac{\partial^2 \Psi_2}{\partial x^2} + V(x, t)\Psi_2 - i\hbar \frac{\partial \Psi_2}{\partial t} \right] \\ &+ c_3 \left[-\frac{\hbar^2}{2m} \frac{\partial^2 \Psi_3}{\partial x^2} + V(x, t)\Psi_3 - i\hbar \frac{\partial \Psi_3}{\partial t} \right] = 0. \end{aligned} \tag{2.4}$$

Equation (2.4) is made up of (2.1), (2.2), and (2.3) (with a few constants thrown in).

$$c_1 \overbrace{\left[-\frac{\hbar^2}{2m} \frac{\partial^2 \Psi_3}{\partial x^2} + V(x, t) \Psi_3 - i\hbar \frac{\partial \Psi_3}{\partial t} \right]}^{(2.1)} + c_2 \overbrace{\left[-\frac{\hbar^2}{2m} \frac{\partial^2 \Psi_3}{\partial x^2} + V(x, t) \Psi_3 - i\hbar \frac{\partial \Psi_3}{\partial t} \right]}^{(2.2)} + c_3 \overbrace{\left[-\frac{\hbar^2}{2m} \frac{\partial^2 \Psi_3}{\partial x^2} + V(x, t) \Psi_3 - i\hbar \frac{\partial \Psi_3}{\partial t} \right]}^{(2.3)} = 0$$

So (2.4) reduces to

$$c_1[0] + c_2[0] + c_3[0] = 0. \quad (2.5)$$

Since (2.5) is certainly true for any values of c_1 , c_2 , c_3 , it follows that $\Psi(x, t)$ is a solution to the Schroedinger equation for a particular potential $V(x, t)$ provided $\Psi_1(x, t)$, $\Psi_2(x, t)$, and $\Psi_3(x, t)$ are solutions to that equation.

3. (From Eisberg & Resnick, P 5-23, pg 172) Consider a particle moving in the potential $V(x)$ plotted in Figure 5-22. For the following ranges of the total energy E , state whether there are any allowed values of E and if so, whether they are discretely separated or continuously distributed. [10]

- (a) $E < V_0$,

Solution: There are no allowed values of E .

- (b) $V_0 < E < V_1$,

Solution: There are allowed values of E , discretely separated since the particle is bound on each side.

- (c) $V_1 < E < V_2$,

Solution: There are allowed values of E , discretely separated since the particle is bound on each side.

- (d) $V_2 < E < V_3$,

Solution: There are allowed values of E , continuously distributed since the particle is free on one side.

- (e) $V_3 < E$.

Solution: There are allowed values of E , continuously distributed since the particle is free on each side.

4. (From Eisberg & Resnick, P 5-27, 5-28, pg 173)

- (a) By substitution into the time-independent Schroedinger equation for the potential illustrated in Figure 5-23, show that in the region to the right of the binding region the eigenfunction has the mathematical form [10]

$$\psi(x) = Ae^{-\frac{\sqrt{2m(V_0-E)}}{\hbar}x} \quad x > +\frac{a}{2}$$

Solution: Figure 5-23 is a well of length a and depth V_0 centred on $x = 0$. Bound states will thus have total energy less than V_0 , or equivalently $(V_0 - E) > 0$. The time-independent Schroedinger equation is

$$-\frac{\hbar^2}{2m} \frac{d^2\psi(x)}{dx^2} + V(x, t)\psi(x) = E\psi(x).$$

For the region to the right of the binding region ($x > a/2$), $V(x) = V_0$, so we can restate the TISE as

$$\frac{d^2\psi(x)}{dx^2} = \frac{2m}{\hbar^2}(V_0 - E)\psi(x). \quad (4.1)$$

Taking the given solution, we can calculate the second derivative

$$\begin{aligned} \frac{d\psi(x)}{dx} &= -\frac{\sqrt{2m(V_0 - E)}}{\hbar} Ae^{-\frac{\sqrt{2m(V_0-E)}}{\hbar}x} \\ \frac{d^2\psi(x)}{dx^2} &= \frac{2m(V_0 - E)}{\hbar^2} Ae^{-\frac{\sqrt{2m(V_0-E)}}{\hbar}x} \\ &= \frac{2m(V_0 - E)}{\hbar^2} \psi(x) \end{aligned} \quad (4.2)$$

Since (4.1) is identical with (4.2) we have shown that the given function is a solution the the TISE.

More generally, since $(V_0 - E) > 0$, the most general solution to (4.1) is of the form

$$\psi(x) = Ae^{-kx} + Be^{kx}, \quad k = \frac{\sqrt{2m(V_0 - E)}}{\hbar}.$$

Since we require that $\psi(x)$ is finite, for $x > a/2$ we must have $B = 0$ otherwise we would have $\psi(x)$ go to infinity as x goes to infinity. This gives us

$$\psi(x) = Ae^{-kx}, \quad k = \frac{\sqrt{2m(V_0 - E)}}{\hbar},$$

which is the same as the given function.

- (b) Using the probability density corresponding to the eigenfunction above, write an expression to estimate the distance D outside the binding region of the potential within which there would be an appreciable probability of finding the particle. [10]

(Hint: Take D to extend to the point at which $\Psi^*\Psi$ is smaller than its value at the edge of the binding region by a factor of e^{-1} . This e^{-1} criterion is similar to one often used in the study of electrical circuits.)

Solution: Note that we are comparing the probability *densities* at different points not the total probability to find the particle in particular regions - we do not need to do any integration.

We would like to find the value of $x' > a/2$ (thus $(2x' - a) > 0$) where

$$\left[\Psi(x, t)^* \Psi(x, t) \right]_{x=x'} = \frac{1}{e} \left[\Psi(x, t)^* \Psi(x, t) \right]_{x=a/2}. \quad (4.3)$$

We also know that $\Psi(x, t) = \psi(x)\phi(t)$ with $\phi(t) = e^{-iEt/\hbar}$ and $\phi(x)$ given above. Evaluating (4.3) gives us:

$$\begin{aligned} \Psi(x', t)^* \Psi(x', t) &= \frac{1}{e} \Psi\left(\frac{a}{2}, t\right)^* \Psi\left(\frac{a}{2}, t\right) \\ \left(A e^{-kx'} e^{+iEt/\hbar} \right) \left(A e^{-kx'} e^{-iEt/\hbar} \right) &= \frac{1}{e} \left(A e^{-ka/2} e^{+iEt/\hbar} \right) \left(A e^{-ka/2} e^{-iEt/\hbar} \right) \\ e^{-2kx'} &= \frac{1}{e} e^{-ka} \\ \frac{1}{e} &= e^{-2kx' + ka} = e^{-k(2x' - a)} \\ e &= e^{k(2x' - a)} \\ \log_e(e) &= \log_e \left(e^{k(2x' - a)} \right) \\ 1 &= k(2x' - a) \implies \frac{1}{k} = (2x' - a) \\ 2x' &= \frac{1}{k} + a \\ x' &= \frac{1}{2k} + \frac{a}{2} \\ x' &= \frac{\hbar}{2\sqrt{2m(V_0 - E)}} + \frac{a}{2} \end{aligned}$$

At the position x' , the probability density will smaller than its value at the edge of the binding region by a factor of e^{-1} . The distance D , measured outside the binding region, is given by $D = x' - a/2$ so we have

$$D = \frac{\hbar}{2\sqrt{2m(V_0 - E)}}.$$

Headstart for next week, Week 10, starting Monday 2004/11/22:

- Read Chapter 6 "Solutions of Time-Independent Schroedinger Equation" in Eisberg & Resnick
- Section 6.1 "Introduction"
- Section 6.2 "The Zero Potential"
- Section 6.3 "The Step Potential' (Energy Less Than Step Height)'
- Section 6.4 "The Step Potential (Energy Greater Than Step Height)''