Physics 202H - Introductory Quantum Physics I
Homework #08 - Solutions

Fall 2004 Due 5:01 PM, Monday 2004/11/15

[55 points total]

"Journal" questions. Briefly share your thoughts on the following questions:
– Of the material that has been covered in the course up to the mid term test, what has been the most difficult for you to understand? What material has been the most interesting? What material has been the most surprising? Is there any material that you thought you understood before this course that you now have a drastically different understanding of? What was is and what has changed?
– Any comments about this week's activities? Course content? Assignment? Lab?

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1. Please complete the anonymous mid-course survey online on WebCT. Early feedback will hopefully allow us to have the best possible course this semester rather than just having next year's students benefit. In addition to the bonus assignment marks, survey participation may count towards overall class participation scores. [5.01-bonus]

   Solution: Do the survey - get the bonus marks.

2. (From Eisberg & Resnick, Q 5-15, pg 168) What is the basic connection between the properties of a wave function and the behaviour of the associated particle? Limit your discussion to about 50 words or so. [10]

   Solution: The wave function determines the probability of finding the associated particle in a given region during a given time period. More specifically, the probability density is given by $P(x,t) = \Psi^*(x,t)\Psi(x,t)$. Not only does the wave function determine the probability density, but allows us to calculate the expectation value of *any* dynamical function of position, time, and momentum of the associated particle (as long as we use the momentum operator in our calculation), thus we can calculate $\overline{x}$, $\overline{x^2}$, $\overline{p}$, etc. From Eisberg & Resnick: "The wave function contains all the information that the uncertainty principle will allow us to learn about the associated particle."

3. (From Eisberg & Resnick, P 5-9, 5-10, 5-11, 5-12, pg 168)

(a) Following the procedure of Example 5-9, verify that the wave function

$$\Psi_2(x,t) = \begin{cases} 0, & x < -a/2, \\ A\sin\left(\frac{2\pi x}{a}\right)e^{-iE_2 t/\hbar}, & -a/2 < x < +a/2, \\ 0, & +a/2 < x. \end{cases}$$

is a solution to the Schroedinger equation in the region $-a/2 < x < +a/2$ for a particle which moves freely through the region but which is strictly confined to it and determine the value of the total energy E_2 of the particle in this excited state of the system, and compare with the total energy of the ground state E_1 found in Example 5-9. [10]

Solution: The Schroedinger equation is

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi(x,t)}{\partial x^2} + V(x,t)\Psi(x,t) = i\hbar\frac{\partial\Psi(x,t)}{\partial t}.$$

For the region between $x > -1/2$ and $x < +a/2$, the potential energy is zero $V(x,t) = 0$, so the Schroedinger equation reduces to

$$\begin{aligned} -\frac{\hbar^2}{2m}\frac{\partial^2\Psi(x,t)}{\partial x^2} &= i\hbar\frac{\partial\Psi(x,t)}{\partial t} \\ \frac{\partial^2\Psi(x,t)}{\partial x^2} &= \frac{-i2m}{\hbar}\frac{\partial\Psi(x,t)}{\partial t} \\ \frac{\partial^2\Psi(x,t)}{\partial x^2} &= -ik^2\frac{\partial\Psi(x,t)}{\partial t}, \qquad k = \sqrt{\frac{2m}{\hbar}}. \end{aligned} \tag{2.1}$$

The various derrivatives of $\Psi_2(x,t)$ in the region $-a/2 < x < +a/2$ are:

$$\frac{\partial\Psi_2(x,t)}{\partial x} = A\frac{2\pi}{a}\cos\left(\frac{2\pi x}{a}\right)e^{-iE_2 t/\hbar}$$

$$\frac{\partial^2\Psi_2(x,t)}{\partial x^2} = -A\frac{4\pi^2}{a^2}\sin\left(\frac{2\pi x}{a}\right)e^{-iE_2 t/\hbar} \tag{2.3}$$

$$\frac{\partial\Psi_2(x,t)}{\partial t} = A\frac{-iE}{\hbar}\sin\left(\frac{2\pi x}{a}\right)e^{-iE_2 t/\hbar} \tag{2.4}$$

Solving (2.3) and (2.4) for like terms and setting them equal to each other gives us

$$\frac{a^2}{4\pi^2}\frac{\partial^2\Psi_2(x,t)}{\partial x^2} = -A\sin\left(\frac{2\pi x}{a}\right)e^{-iE_2 t/\hbar} = \frac{\hbar}{iE_2}\frac{\partial\Psi_2(x,t)}{\partial t}$$

$$\frac{\partial^2\Psi_2(x,t)}{\partial x^2} = -i\frac{4\pi^2\hbar}{a^2E_2}\frac{\partial\Psi_2(x,t)}{\partial t} \tag{2.5}$$

Comparing (2.1) and (2.5), we see that $\Psi_2(x,t)$ is a solution to the Schroedinger equation in the region $-a/2 < x < +a/2$ provided that

$$\frac{2m}{\hbar} = \frac{4\pi^2\hbar}{a^2E_2} \quad\Longrightarrow\quad E_2 = \frac{4\pi^2\hbar^2}{2ma^2} = \frac{2\pi^2\hbar^2}{ma^2} = \frac{h^2}{2ma^2}. \tag{2.6}$$

Thus the given wave function $\Psi_2(x,t)$ is a solution to the Schroedinger equation for $E_2 = (h^2)/(2ma^2)$. Equation (2.6) shows us that $E_2 = (4\pi^2\hbar^2)/(2ma^2)$, while Example 5-9 gave us $E_1 = (\pi^2\hbar^2)/(2ma^2)$, thus $E_2 = 4E_1$.

(b) Plot the space dependence of this wave function $\Psi_2(x,t)$. Compare with the ground state wave function $\Psi_1(x,t)$ of Figure 5-7, and give a qualitative argument relating the difference in the two wave functions to the difference in the total energies of the two states. [5]

Solution: The wave function of the ground state $\Psi_1(x,t)$ has a space dependence which is one half of a complete sin cycle. In figure 1 we have plotted the normalized wave functions, anticipating the result of the next problem, with $a = 1$. We used Maple with the following commands to generate the plot.

```
> restart; with(plots);
> plot([(sqrt(2)*cos(Pi*x)),(sqrt(2)*sin(2*Pi*x))], x=-(0.5)..(0.5),
  colour=[navy, blue], legend=["psi_1", "psi_2"], linestyle=[3,2]);
```

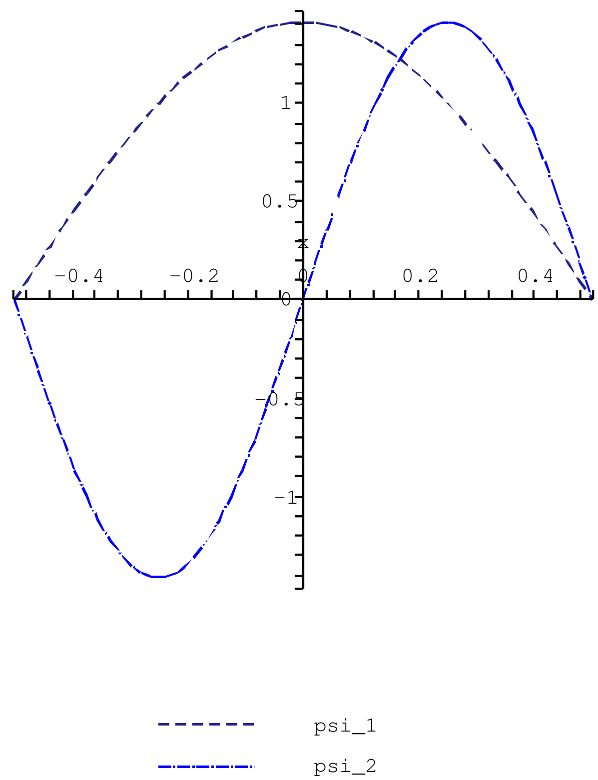

Figure 1: $\psi_1(x)$ and $\psi_2(x)$ with $a = 1$

As we can see in figure 1, the wave function of the excited state $\Psi_2(x,t)$ has a space dependence which is a complete sin cycle. Thus the wavelength of $\Psi_1(x,t)$ is twice that of $\Psi_2(x,t)$, which as we expect, corresponds with a greater energy for $\Psi_2(x,t)$ compared with $\Psi_1(x,t)$. $\Psi_2(x,t)$ also has a more sharply curved shape, also corresponding to having greater energy.

(c) Normalize the wave function $\Psi_2(x,t)$ above by adjusting the value of the multiplicative constant A so that the total probability of finding the associated particle somewhere in the region of length a equals one. Compare with the value of A obtained in Example 5-10 by normalizing the ground state wave function $\Psi_1(x,t)$. Discuss the comparison. [10]

Solution: To normalize a wave function we integrate $P(x,t)\mathrm{d}x = \Psi^*(x,t)\Psi(x,t)\mathrm{d}x$ over all values of x, and set that equal to one. For the wave function in question, it has a value of zero for any x outside of $|x| < a/2$ so:

$$
\begin{aligned}
1 &= \int_{-\infty}^{+\infty} P(x,t)\mathrm{d}x = \int_{-a/2}^{+a/2} \Psi_2^*(x,t)\Psi_2(x,t)\mathrm{d}x \\
&= \int_{-a/2}^{+a/2} A\sin\left(\frac{2\pi x}{a}\right)\mathrm{e}^{\mathrm{i}Et/\hbar} A\sin\left(\frac{2\pi x}{a}\right)\mathrm{e}^{-\mathrm{i}Et/\hbar}\mathrm{d}x \\
&= A^2\int_{-a/2}^{+a/2} \sin^2\left(\frac{2\pi x}{a}\right)\mathrm{d}x \qquad u = \frac{2\pi x}{a}, \mathrm{d}u = \frac{2\pi}{a}\mathrm{d}x \\
&= A^2\frac{a}{2\pi}\int_{-\pi}^{+\pi} \sin^2 u\,\mathrm{d}u = A^2\frac{a}{2\pi}\left[\frac{u}{2} - \frac{\sin(2u)}{4}\right]_{-\pi}^{+\pi} = A^2\frac{a}{2\pi}\left[\frac{\pi}{2} + \frac{\pi}{2}\right] \\
1 &= A^2\frac{a}{2} \quad \Longrightarrow \quad A = \sqrt{\frac{2}{a}}.
\end{aligned}
$$

With this value for A, the complete normalized wave function is

$$
\Psi_2(x,t) = \begin{cases} 0, & x < -a/2, \\ \sqrt{\frac{2}{a}}\sin\left(\frac{2\pi x}{a}\right)\mathrm{e}^{-\mathrm{i}Et/\hbar}, & -a/2 < x < +a/2, \\ 0, & +a/2 < x. \end{cases}
$$

The normalization for $\Psi_1(x,t)$ is identical with the normalization for $\Psi_2(x,t)$, in each case $A = \sqrt{2/a}$.

$$
\Psi_1(x,t) = \begin{cases} 0, & x < -a/2, \\ \sqrt{\frac{2}{a}}\cos\left(\frac{\pi x}{a}\right)\mathrm{e}^{-\mathrm{i}Et/\hbar}, & -a/2 < x < +a/2, \\ 0, & +a/2 < x. \end{cases}
$$

(d) Calculate the expectation value of x, the expectation value of x^2, the expectation value of p, and the expectation value of p^2 for the particle associated with the wave function $\Psi_2(x,t)$ above. [10]

Solution: The expectation value of x is calculated by integrating $\Psi_2^* x\Psi_2\mathrm{d}x$ and the expectation value of x^2 is calculated by integrating $\Psi_2^* x^2\Psi_2\mathrm{d}x$.

$$
\begin{aligned}
\overline{x} &= \int_{-a/2}^{+a/2} \Psi^*(x,t)x\Psi(x,t)\mathrm{d}x \\
&= \int_{-a/2}^{+a/2} A\sin\left(\frac{2\pi x}{a}\right)\mathrm{e}^{\mathrm{i}Et/\hbar} xA\sin\left(\frac{2\pi x}{a}\right)\mathrm{e}^{-\mathrm{i}Et/\hbar}\mathrm{d}x \\
&= \frac{2}{a}\int_{-a/2}^{+a/2} x\sin^2\left(\frac{2\pi x}{a}\right)\mathrm{d}x \qquad u = \frac{2\pi x}{a}, \mathrm{d}u = \frac{2\pi}{a}\mathrm{d}x \\
&= \frac{a}{2\pi^2}\int_{-\pi}^{+\pi} \underbrace{\overbrace{u}^{\text{odd}}\overbrace{\sin^2 u}^{\text{even}}}_{\text{odd}}\,\mathrm{d}u = 0 \\
\overline{x} &= 0.
\end{aligned}
$$

$$
\begin{aligned}
\overline{x^2} &= \int_{-a/2}^{+a/2} \Psi^*(x,t) x^2 \Psi(x,t) \mathrm{d}x \\
&= \int_{-a/2}^{+a/2} A \sin\left(\frac{2\pi x}{a}\right) \mathrm{e}^{\mathrm{i}Et/\hbar} x^2 A \sin\left(\frac{2\pi x}{a}\right) \mathrm{e}^{-\mathrm{i}Et/\hbar} \mathrm{d}x \\
&= \frac{2}{a} \int_{-a/2}^{+a/2} x^2 \sin^2\left(\frac{2\pi x}{a}\right) \mathrm{d}x \qquad u = \frac{2\pi x}{a}, \mathrm{d}u = \frac{2\pi}{a} \mathrm{d}x \\
&= \frac{a^2}{4\pi^3} \int_{-\pi}^{+\pi} \underbrace{u^2 \sin^2 u}_{\text{even}} \, \mathrm{d}u = \frac{a^2}{2\pi^3} \int_0^{+\pi} u^2 \sin^2 u \, \mathrm{d}u
\end{aligned}
$$

We can use the trig identity $\sin^2 u = (1 - \cos(2u))/2$ and the integral formula $\int u^2 \cos(2u) \, \mathrm{d}u = (u/2)\cos(2u) + (u^2/2 - 1/4)\sin(2u)$ to simplify things a little bit:

$$
\begin{aligned}
\overline{x^2} &= \frac{a^2}{2\pi^3} \int_0^{+\pi} u^2 \frac{1 - \cos(2u)}{2} \, \mathrm{d}u \\
&= \frac{a^2}{4\pi^3} \int_0^{+\pi} (u^2 - u^2 \cos(2u)) \, \mathrm{d}u \\
&= \frac{a^2}{4\pi^3} \left[\frac{u^3}{3} - \frac{u}{2} \cos(2u) - \left(\frac{u^2}{2} - \frac{1}{4} \right) \sin(2u) \right]_0^{+\pi} \\
&= \frac{a^2}{4\pi^3} \left[\frac{\pi^3}{3} - \frac{\pi}{2} \right] = a^2 \left(\frac{1}{12} - \frac{1}{8\pi^2} \right) \\
\overline{x^2} &\approx (0.07066\ldots) a^2
\end{aligned}
$$

To calculate the expectation value for momentum, we need to use the momentum operator $-\mathrm{i}\hbar(\partial/\partial x)$.

$$
\begin{aligned}
\overline{p} &= \int_{-a/2}^{+a/2} \Psi^*(x,t) \left(-\mathrm{i}\hbar \frac{\partial}{\partial x} \right) \Psi(x,t) \mathrm{d}x \\
&= \int_{-a/2}^{+a/2} A \sin\left(\frac{2\pi x}{a}\right) \mathrm{e}^{\mathrm{i}Et/\hbar} \left(-\mathrm{i}\hbar \frac{\partial}{\partial x} \right) A \sin\left(\frac{2\pi x}{a}\right) \mathrm{e}^{-\mathrm{i}Et/\hbar} \mathrm{d}x \\
&= \frac{2}{a} \int_{-a/2}^{+a/2} \sin\left(\frac{2\pi x}{a}\right) \frac{-\mathrm{i}\hbar 2\pi}{a} \cos\left(\frac{2\pi x}{a}\right) \mathrm{d}x \qquad u = \frac{2\pi x}{a}, \mathrm{d}u = \frac{2\pi}{a} \mathrm{d}x \\
&= \frac{-\mathrm{i}\hbar}{\pi} \int_{-\pi}^{+\pi} \underbrace{\overbrace{\sin u}^{\text{odd}} \overbrace{\cos u}^{\text{even}}}_{\text{odd}} \, \mathrm{d}u = 0 \\
\overline{p} &= 0.
\end{aligned}
$$

To calculate the expectation value for p^2 we need to use $-\hbar^2(\partial^2/\partial x^2)$.

$$
\begin{aligned}
\overline{p^2} &= \int_{-a/2}^{+a/2} \Psi^*(x,t)\left(-\hbar^2\frac{\partial^2}{\partial x^2}\right)\Psi(x,t)\mathrm{d}x \\
&= \int_{-a/2}^{+a/2} A\sin\left(\frac{2\pi x}{a}\right)\mathrm{e}^{\mathrm{i}Et/\hbar}\left(-\hbar^2\frac{\partial^2}{\partial x^2}\right)A\sin\left(\frac{2\pi x}{a}\right)\mathrm{e}^{-\mathrm{i}Et/\hbar}\mathrm{d}x \\
&= \frac{2}{a}\int_{-a/2}^{+a/2} -\hbar^2\left(\frac{-4\pi^2}{a^2}\right)\sin^2\left(\frac{2\pi x}{a}\right)\mathrm{d}x \\
&= \frac{8\pi^2\hbar^2}{a^3}\int_{-a/2}^{+a/2}\sin^2\left(\frac{2\pi x}{a}\right)\mathrm{d}x \qquad u=\frac{2\pi x}{a},\ \mathrm{d}u=\frac{2\pi}{a}\mathrm{d}x \\
&= \frac{4\pi\hbar^2}{a^2}\int_{-\pi}^{+\pi}\sin^2 u\,\mathrm{d}u = \frac{4\pi\hbar^2}{a^2}\left[\frac{u}{2}-\frac{\sin(2u)}{4}\right]_{-\pi}^{+\pi} = \frac{4\pi\hbar^2}{a^2}\left[\frac{\pi}{2}+\frac{\pi}{2}\right] \\
\overline{p^2} &= \frac{4\pi^2\hbar^2}{a^2} = \left(\frac{2\pi\hbar}{a}\right)^2 = \frac{h^2}{a^2}
\end{aligned}
$$

The expectation value of x is zero, and the expectation value of x^2 is about $(0.0707)a^2$. The expectation value of p is zero, and the expectation value of p^2 is h^2/a^2. Note that the square root of the expectation value of p^2, namely $\sqrt{\overline{p^2}} = h/a$ (called the root-mean-square value of p) is a measure of the fluctuations around the average $\overline{p} = 0$. This RMS value is also a measure of the uncertainty in p.

4. (From Eisberg & Resnick, P 5-16, pg 169) Show by direct substitution into the Schroedinger equation that the wave function $\Psi(x,t) = \psi(x)\mathrm{e}^{-\mathrm{i}Et/\hbar}$ satisfies that equation if the eigenfunction $\psi(x)$ satisfies the time-independent Schroedinger equation for a potential $V(x)$. [10]

Solution: The Schroedinger equation is

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi(x,t)}{\partial x^2} + V(x,t)\Psi(x,t) = \mathrm{i}\hbar\frac{\partial\Psi(x,t)}{\partial t}. \tag{4.1}$$

If we make the substitution $\Psi(x,t) = \psi(x)\mathrm{e}^{-\mathrm{i}Et/\hbar}$ and $V(x,t) = V(x)$ we have

$$\begin{aligned} -\frac{\hbar^2}{2m}\frac{\partial^2\left[\psi(x)\mathrm{e}^{-\mathrm{i}Et/\hbar}\right]}{\partial x^2} + V(x,t)\left[\psi(x)\mathrm{e}^{-\mathrm{i}Et/\hbar}\right] &= \mathrm{i}\hbar\frac{\partial\left[\psi(x)\mathrm{e}^{-\mathrm{i}Et/\hbar}\right]}{\partial t} \\ -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\psi(x)}{\mathrm{d}x^2}\mathrm{e}^{-\mathrm{i}Et/\hbar} + V(x,t)\psi(x)\mathrm{e}^{-\mathrm{i}Et/\hbar} &= \mathrm{i}\hbar\psi(x)\frac{\mathrm{d}\mathrm{e}^{-\mathrm{i}Et/\hbar}}{\mathrm{d}t} \\ -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\psi(x)}{\mathrm{d}x^2}\mathrm{e}^{-\mathrm{i}Et/\hbar} + V(x,t)\psi(x)\mathrm{e}^{-\mathrm{i}Et/\hbar} &= \mathrm{i}\hbar\psi(x)\frac{-\mathrm{i}E}{\hbar}\mathrm{e}^{-\mathrm{i}Et/\hbar} \\ \left(-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\psi(x)}{\mathrm{d}x^2} + V(x,t)\psi(x)\right)\mathrm{e}^{-\mathrm{i}Et/\hbar} &= \left(E\psi(x)\right)\mathrm{e}^{-\mathrm{i}Et/\hbar}. \end{aligned} \tag{4.2}$$

We can divide (4.2) through by $\mathrm{e}^{-\mathrm{i}Et/\hbar}$ to get

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\psi(x)}{\mathrm{d}x^2} + V(x,t)\psi(x) = E\psi(x). \tag{4.3}$$

If $\psi(x)$ satisfies (4.3), then working upwards from line (4.3) we have that $\Psi(x,t)$ satisfies the Schroedinger equation (4.1).

Equation (4.3) is the time-independent Schroedinger equation for a potential $V(x)$, and we were given that $\psi(x)$ is a solution to the time-independent Schroedinger equation for a potential $V(x)$, therefore, $\Psi(x,t)$ is a solution to the Schroedinger equation.

Headstart for next week, Week 09, starting Monday 2004/11/15:
– Read Chapter 5 “Schroedinger’s Theory of Quantum Mechanics” in Eisberg & Resnick
– – Section 5.6 “Required properties of Eigenfunctions”
– – Section 5.7 “Energy Quantization in the Schroedinger Theory”
– – Section 5.8 “Summary”