Physics 202H - Introductory Quantum Physics I
Homework #07 - Solutions

Fall 2004 Due 5:01 PM, Monday 2004/11/08

[80 points total]

"Journal" questions. Briefly share your thoughts on the following questions:
– What physics material do you recall from your elementary school experiences (up to about age 12)? How was it presented? What was your perception of the instructor's attitude to the material? What about other non-physics sciences? Math?
– Any comments about this week's activities? Course content? Assignment? Lab?

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1. (From Eisberg & Resnick, P A-11, pg A-18) With modifications.
   (a) What potential difference will accelerate an electron to the speed of light according to classical (non-relativistic) physics? [5]

   Solution: The particle's kinetic energy is $K = q\Delta V = eV$ where V is the electrical potential difference it is accelerated through. According to classical physics the kinetic energy is also given by $K = mv^2/2$, thus

$$
\begin{aligned}
K &= eV = \frac{1}{2}mv^2 \\
V &= \frac{mv^2}{2e} = \frac{m_e c^2}{2e} = \frac{(9.109\ldots \times 10^{-31}\,\mathrm{kg})(299792458\,\mathrm{m/s})^2}{2(1.6021\ldots \times 10^{-19}\,\mathrm{Coul})} \\
&= 255499.5\ldots\ \mathrm{V} \approx 255\,\mathrm{kV} = 0.255\,\mathrm{MV}
\end{aligned}
$$

   Classically, about 255 kV or 0.255 MV or 2.55×10^5 V would accelerate an electron to the speed of light.

   (b) With this potential difference, what speed will an electron acquire relativistically? [5]

   Solution: With this electric potential difference, the electron would have the same kinetic energy as above $K = eV = \frac{1}{2}m_e c^2 \approx 0.255\,\mathrm{MeV} \approx 4.09 \times 10^{-14}\,\mathrm{J}$, but now we need to use the relativistic relationships to find the electron's speed, $K = eV = E - m_0c^2$, $E^2 = p^2c^2 + m_0^2c^4$ and $p = \gamma m_0 v$,

$$
\begin{aligned}
E^2 = (K + m_e c^2)^2 &= p^2c^2 + m_e^2c^4 \\
p^2c^2 &= (K + m_e c^2)^2 - m_e^2c^4 \\
p^2c^2 &= K^2 + 2Km_e c^2 + m_e^2c^4 - m_e^2c^4 \\
&= K^2 + 2Km_e c^2 \\
\gamma^2 m_e^2 v^2 c^2 &= K^2 + 2Km_e c^2 \\
\frac{v^2}{1 - \frac{v^2}{c^2}} &= \frac{K^2 + 2Km_e c^2}{m_e^2 c^2} \\
\frac{v^2}{c^2 - v^2} &= \frac{K^2 + 2Km_e c^2}{m_e^2 c^4}.
\end{aligned}
$$

If we let the right hand side of this relationship be A, we can solve for v^2 to get

$$\frac{v^2}{c^2 - v^2} = A \implies v^2 = Ac^2 - Av^2$$
$$v^2 + Av^2 = Ac^2 \implies v^2(1 + A) = Ac^2$$
$$v^2 = c^2 \frac{A}{1 + A} = c^2 \frac{\frac{K^2 + 2Km_e c^2}{m_e^2 c^4}}{1 + \frac{K^2 + 2Km_e c^2}{m_e^2 c^4}}$$
$$= c^2 \frac{(K^2 + 2Km_e c^2)}{m_e^2 c^4 + K^2 + 2Km_e c^2}$$
$$= c^2 \frac{(K^2 + 2Km_e c^2)}{(K + m_e c^2)^2}.$$

With $K = eV = \frac{1}{2} m_e c^2 \approx 0.255\,\text{MeV} \approx 4.09 \times 10^{-14}\,\text{J}$ we have

$$v^2 = c^2 \frac{(K^2 + 2Km_e c^2)}{(K + m_e c^2)^2}$$
$$= c^2 \frac{(\frac{1}{2} m_e c^2)^2 + 2(\frac{1}{2} m_e c^2)(m_e c^2)}{((\frac{1}{2} m_e c^2) + m_e c^2)^2}$$
$$= c^2 \frac{\frac{5}{4} m_e^2 c^4}{\frac{9}{4} m_e^2 c^4} = \frac{5}{9} c^2$$
$$v = \frac{\sqrt{5}}{3} c = (0.74535\ldots)c = 2.234521\ldots \times 10^8\,\text{m/s}$$
$$\approx (0.745)c \approx 2.23 \times 10^8\,\text{m/s}.$$

An electron accelerated through this electrical potential difference will move at about 74.5% of the speed of light or about 2.23×10^8 m/s.

(c) What would its relativistic momentum be at this speed? [5]

Solution: From above, we have an expression for the relativistic momentum:

$$p^2 c^2 = K^2 + 2Km_e c^2$$
$$= (\frac{1}{2} m_e c^2)^2 + 2(\frac{1}{2} m_e c^2)(m_e c^2)$$
$$= \frac{5}{4} m_e^2 c^4$$
$$pc = \frac{\sqrt{5}}{2} m_e c^2 = (1.1180339887)(0.511\,\text{MeV}) = 0.57131\,\text{MeV} = 9.1533 \times 10^{-14}\,\text{J}$$
$$p = \frac{\sqrt{5}}{2} m_e c = 3.0532\ldots \times 10^{-22}\,\text{kg} \cdot \text{m/s}$$

The relativistic momentum of the electron would be about 3.05×10^{-22} kg · m/s.

(d) What would its relativistic kinetic energy be at this speed? [5]

Solution: The relativistic kinetic energy is the same as the non-relativistic kinetic energy as calculated above, $K = eV = \frac{1}{2} m_e c^2 \approx 0.255\,\text{MeV} \approx 4.09 \times 10^{-14}\,\text{J}$.

2. (From Eisberg & Resnick, P 1-9, pg 23) Assuming that $\lambda_{\max}$ is in the near infrared for red heat and in the near ultraviolet for blue heat, approximately what temperature in Wien's displacement law corresponds to red heat? To blue heat? [10]

   Solution: Wien's displacement law gives us that $\lambda_{\max} T = \text{const} = 2.898 \times 10^{-3}\,\text{m}\cdot{}^\circ\text{K}$. Depending on the source one looks at visible light is between about $400 - 700\,\text{nm}$, near infrared wavelengths range from about $770 - 1400\,\text{nm}$ (or more depending on who is defining the range) and near ultraviolet wavelengths range from about $200 - 380\,\text{nm}$. Thus a good typical wavelength for NIR is $\lambda_{\text{NIR}} = 10^{-6}\,\text{m}$ and for NUV is $\lambda_{\text{NUV}} = 3 \times 10^{-7}\,\text{m}$.

$$\begin{aligned} T_{\text{red}} &= \frac{2.898 \times 10^{-3}\,\text{m}\cdot{}^\circ\text{K}}{\lambda_{\text{NIR}}} \\ &= \frac{2.898 \times 10^{-3}\,\text{m}\cdot{}^\circ\text{K}}{10^{-6}\,\text{m}} \\ &= 2898\,{}^\circ\text{K} \\ &\approx 2900\,{}^\circ\text{K} \end{aligned} \qquad \begin{aligned} T_{\text{blue}} &= \frac{2.898 \times 10^{-3}\,\text{m}\cdot{}^\circ\text{K}}{\lambda_{\text{NUV}}} \\ &= \frac{2.898 \times 10^{-3}\,\text{m}\cdot{}^\circ\text{K}}{3 \times 10^{-7}\,\text{m}} \\ &= 9660\,{}^\circ\text{K} \\ &\approx 9700\,{}^\circ\text{K} \end{aligned}$$

   "Red heat" is approximately 2900 °K and "blue heat" is about 9700 °K.

3. (From Eisberg & Resnick, P 2-20, pg 53) What fractional increase in wavelength leads to a 75% loss of photon energy in a Compton collision? [10]

   Solution: Before scattering the photon has an energy of $E_i = hc/\lambda_i$ and after scattering it has an energy of $E_f = hc/\lambda_f$ where $\lambda_f = \lambda_i + \Delta\lambda$. The fractional loss of energy is thus

$$\begin{aligned} f &= \frac{E_i - E_f}{E_i} = \frac{\frac{hc}{\lambda_i} - \frac{hc}{\lambda_f}}{\frac{hc}{\lambda_i}} \\ &= \frac{1 - \frac{\lambda_i}{\lambda_f}}{1} = \frac{\lambda_f - \lambda_i}{\lambda_f} = \frac{(\lambda_i + \Delta\lambda) - \lambda_i}{(\lambda_i + \Delta\lambda)} \\ f &= \frac{\Delta\lambda}{\Delta\lambda + \lambda_i}. \end{aligned}$$

   If $f = 0.75$ then we can find the fractional increase in wavelength of $\Delta\lambda/\lambda_i$ via

$$\begin{aligned} f = \frac{\Delta\lambda}{\Delta\lambda + \lambda_i} \quad \implies \quad f\Delta\lambda + f\lambda_i &= \Delta\lambda \\ f\lambda_i &= \Delta\lambda - f\Delta\lambda = \Delta\lambda(1 - f) \\ \frac{f}{1 - f} &= \frac{\Delta\lambda}{\lambda_i} \\ \frac{\Delta\lambda}{\lambda_i} &= \frac{f}{1 - f} \\ &= \frac{\frac{3}{4}}{1 - \frac{3}{4}} \\ \frac{\Delta\lambda}{\lambda_i} &= 3 \end{aligned}$$

   So a fractional increase in wavelength of 300% leads to a 75% loss of photon energy in a Compton collision. Note that the final wavelength is four times (400%) the initial wavelength, $\lambda_f = 4\lambda_i$, but the *fractional* increase is 300%.

4. (From Eisberg & Resnick, P 2-31, pg 54) An electron-positron pair at rest annihilate, creating two photons. At what speed must an observer move along the line of the photons in order that the wavelength of one photon be twice that of the other? [10]

Solution: In the centre of mass rest frame, the energy of the two photons must be equal to the rest mass energy of the electron-positron pair, so each photon will have an energy of $E_{photon} = m_e c^2$, and a wavelength of $\lambda = hc/E_{photon} = h/m_e c$. For an observer moving with respect to the centre of mass rest frame at velocity v, the relativistic Doppler shift allows us to calculate the wavelengths of the two photons which moving in opposite directions. See Eisberg & Resnick, Example 2-7, pg 46 for an example of this in detail.

$$\lambda_1 = \lambda\sqrt{\frac{c-v}{c+v}} = \frac{h}{m_e c}\sqrt{\frac{c-v}{c+v}} \qquad \lambda_2 = \lambda\sqrt{\frac{c+v}{c-v}} = \frac{h}{m_e c}\sqrt{\frac{c+v}{c-v}}$$

$$\begin{aligned}
\lambda_2 = 2\lambda_1 \quad \implies \quad \frac{h}{m_e c}\sqrt{\frac{c+v}{c-v}} &= 2\frac{h}{m_e c}\sqrt{\frac{c-v}{c+v}} \\
\sqrt{\frac{c+v}{c-v}} &= 2\sqrt{\frac{c-v}{c+v}} \\
c+v &= 2(c-v) \\
3v &= c \\
v &= \frac{c}{3} = (9.9930\ldots \times 10^7 \text{ m/s}) \\
v &= \frac{c}{3} \approx 10^8 \text{ m/s}
\end{aligned}$$

The observer would have to move at $c/3$ or about 10^8 m/s for one of the photons to have twice the wavelength of the other photon. Note that if we had said that $\lambda_1 = 2\lambda_2$, we would have got a result that $v = -c/3$, indicating that the observer would have to move in the other direction.

We cannot simply do a relativistic length contraction of the wavelength of the photon since the wavelength of the photon is closely related to the frequency of the photon and the speed of light - it is not just a rod of length λ that we can contract - we must necessarily use both length and time transformations for each photon. If you just used length contraction, each wavelength would have the same contraction - they would have equal wavelengths (to each other) in the observer's frame. Rather than doing the correct time/length calculation, it is much simpler to use the energy and momentum conservation as done in Eisberg & Resnick.

5. (From Eisberg & Resnick, P 3-2, pg 81) The wavelength of the yellow spectral emission of sodium is 5890 Å. At what kinetic energy would an electron have the same de Broglie wavelength? [10]

Solution: We want to find the kinetic energy of an electron so that it has a wavelength of $\lambda = 5890$ Å. Since $K = \frac{1}{2}mv^2 = p^2/2m$ we have $p = \sqrt{2mK}$ and

$$\begin{aligned}
\lambda &= \frac{h}{p} = \frac{h}{\sqrt{2mK}} \\
2mK &= \frac{h^2}{\lambda^2} \\
K &= \frac{h^2}{2m\lambda^2} = \frac{(6.6260 \times 10^{-34}\,\mathrm{J\cdot s})^2}{2(9.109 \times 10^{-31}\,\mathrm{kg})(5890\ \times 10^{-10}\,\mathrm{m})^2} \\
&= 6.94644\ldots \times 10^{-25}\,\mathrm{J} = 4.33563\ldots\ \times 10^{-6}\,\mathrm{eV} \\
K &\approx 6.95\ \times 10^{-25}\,\mathrm{J} \approx 4.34\ \times 10^{-6}\,\mathrm{eV}.
\end{aligned}$$

An electron with kinetic energy of about $6.95\ \times 10^{-25}$ J or $4.34\ \times 10^{-6}$ eV would have a de Broglie wavelength equal to the wavelength of the yellow spectral emission of sodium. Note that this is a very small energy, but it still corresponds with a velocity of about 1230 m/s. Since the kinetic energy is much less than the rest mass energy of the electron, our non-relativistic calculation is accurate. Equivalently, the speed of 1230 m/s is much less than the speed of light c, so again the non-relativistic calculation is justified.

6. (From Eisberg & Resnick, P 3-27, pg 83) The velocity of a positron is measured to be: $v_x = (4.00 \pm 0.18) \times 10^5$ m/s, $v_y = (0.34 \pm 0.12) \times 10^5$ m/s, $v_z = (1.41 \pm 0.08) \times 10^5$ m/s. Within what minimum volume was the positron located at the moment the measurement was carried out? (Note: I think that the answer in the back of the text is incorrect for this problem - it is out by a factor of about 16) [10]

Solution: We are given the uncertainties in the three components of the velocity vector, and from that can calculate the uncertainties for the momentum vector, $\Delta\vec{p} = m\Delta\vec{v}$. Heisenberg's uncertainty relationship will then let us place limits on the uncertainty in the position vector. Given the minimum uncertainty in position, we can calculate the volume of the box that the positron will be located in.

$$\begin{aligned}
\Delta v_x &= 0.18 \times 10^5\,\mathrm{m/s} & \Delta v_y &= 0.12 \times 10^5\,\mathrm{m/s} & \Delta v_z &= 0.08 \times 10^5\,\mathrm{m/s} \\
\Delta p_x &= m\Delta v_x & \Delta p_y &= m\Delta v_y & \Delta p_z &= m\Delta v_z \\
\Delta x &> \frac{\hbar}{2\Delta p_x} = \frac{\hbar}{2m\Delta v_x} & \Delta y &> \frac{\hbar}{2m\Delta v_y} & \Delta z &> \frac{\hbar}{2m\Delta v_z}
\end{aligned}$$

Since each of these uncertainties in position represents the maximum deviation from the value of the position, both positive and negative, the minimum volume of the box in question is given by multiplying $2\Delta x$ by $2\Delta y$ by $2\Delta z$. Thus we have

$$\begin{aligned}
V &= (2\Delta x)(2\Delta y)(2\Delta z) > 2\left(\frac{\hbar}{2m\Delta v_x}\right)2\left(\frac{\hbar}{2m\Delta v_y}\right)2\left(\frac{\hbar}{2m\Delta v_z}\right) \\
&> \frac{\hbar^3}{m^3\Delta v_x\Delta v_y\Delta v_z} = \frac{(1.05457148 \times 10^{-34}\,\mathrm{J\cdot s})^3}{(9.10938188 \times 10^{-31}\,\mathrm{kg})^3(0.18)(0.12)(0.08)(10^5\,\mathrm{m/s})^3} \\
V &> 8.97879\ldots \times 10^{-25}\,\mathrm{m}^3 \approx 8.98 \times 10^{-25}\,\mathrm{m}^3.
\end{aligned}$$

At the moment when the measurement of the particle's velocity was carried out, it was located somewhere within a box who's volume was about $8.98 \times 10^{-25}\,\mathrm{m}^3$. Forgetting to take into account the $\pm$ nature of the individual uncertainties in position would result in an answer that was $2^3 = 8$ times smaller than that calculated above.

Eisberg & Resnick get a different answer for this question (see pg S-1). They seem to have taken the uncertainties in velocity as being twice what I understand the uncertainties to be. They then use these values to find the minimum uncertainty in position as above

$$\Delta v'_x = 0.36 \times 10^5\,\text{m/s} \qquad \Delta v'_y = 0.24 \times 10^5\,\text{m/s} \qquad \Delta v'_z = 0.16 \times 10^5\,\text{m/s}$$
$$\Delta p'_x = m\Delta v'_x \qquad \Delta p'_y = m\Delta v'_y \qquad \Delta p'_z = m\Delta v'_z$$
$$\Delta x' > \frac{\hbar}{2\Delta p'_x} = \frac{\hbar}{2m\Delta v'_x} \qquad \Delta y' > \frac{\hbar}{2m\Delta v'_y} \qquad \Delta z' > \frac{\hbar}{2m\Delta v'_z}$$

To calculate the minimum volume, Eisberg & Resnick seem to have just multiplied $\Delta x'$ by $\Delta y'$ by $\Delta z'$, with no factors of two to take into account the $\pm$ nature of the uncertainty in position.

$$\begin{aligned} V' = (\Delta x')(\Delta y')(\Delta z') &> \left(\frac{\hbar}{2m\Delta v'_x}\right)\left(\frac{\hbar}{2m\Delta v'_y}\right)\left(\frac{\hbar}{2m\Delta v'_z}\right) \\ &> \frac{\hbar^3}{8m^3\Delta v'_x\Delta v'_y\Delta v'_z} = \frac{(1.05457148 \times 10^{-34}\,\text{J}\cdot\text{s})^3}{8(9.10938188 \times 10^{-31}\,\text{kg})^3(0.36)(0.24)(0.16)(10^5\,\text{m/s})^3} \\ V' &> 1.4029\ldots \times 10^{-26}\,\text{m}^3 \approx 1.40 \times 10^4\,\text{Å}^3. \end{aligned}$$

The uncertainties Δx and Δp_x in the Heisenberg uncertainty relationship clearly represent the standard deviation from the expectation value of the position and momentum of the particle (and this notation is consistent with standard methods of reporting uncertainty). Thus in general you would expect to use $2\Delta x$ when describing the range of values of position (as we did when calculating the minimum volume of the box), but *not* use $2\Delta p_x = m2\Delta v_x$ in the Heisenberg relation in order to calculate the range for Δx, as Eisberg & Resnick seem to have done.

7. (From Eisberg & Resnick, Q 5-5, pg 168) Why is the Schroedinger equation not valid for relativistic particles? Limit your discussion to about 50 words or so. [10]

Solution: The Schroedinger equation was created using the non-relativistic expressions for momentum ($p = mv$) and kinetic energy ($k = \frac{1}{2}mv^2$). Most importantly the expression for total energy used to develop the Schroedinger, $E = p^2/2m + V$, is non-relativistic. Since the starting point for the entire structure of the Schroedinger equation was non-relativistic, it should not be too surprising that the results are non-relativistic as well. It was not until Dirac's work in 1929 that a relativistic form of quantum mechanics was put together.

Headstart for next week, Week 08, starting Monday 2004/11/08:
– Review Section 2.3.6 "The Schrödinger equation" in "Simple Nature" by Crowellk
– Read Chapter 5 "Schroedinger's Theory of Quantum Mechanics" in Eisberg & Resnick
– – Section 5.4 "Expectation Values"
– – Section 5.5 "The Time-Independent Schroedinger Equation"