Physics 202H - Introductory Quantum Physics I
Homework #05 - Solutions

Fall 2004 Due 5:01 PM, Monday 2004/10/18

[55 points total]

"Journal" questions. Briefly share your thoughts on the following questions:
– Have you ever noticed any physics (or science or math or technology if you cannot recall a physics example) issue/idea/result presented incorrectly in the general media or popular press? In a non-science course? What was it? What, if anything, should be done about this type of problem? Is it a problem? Why or why not?
– Any comments about this week's activities? Course content? Assignment? Lab?

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1. (From Eisberg & Resnick, Q 4-3, pg 119) List objections to the Thomson model of the atom. Limit your discussion to about 50 words or so. [10]

   Solution: The Thompson model (the "plum pudding" model) has a number of things that are objectionable. The Thompson model predicts a single characteristic emission frequency for each elemental atom, whereas observations show a very large number of different frequencies observed in the spectrum of hydrogen and other elements (the Rutherford model does not really improve on this, but the Bohr model does a very respectable job). Most problematic however is that is predicts essentially zero scattering of alpha particles at large angles, while experiments reveal that there are a few such scattering events, but many many many more than the Thompson model predicts (the Rutherford model predicts these scattering events very well).

2. (From Eisberg & Resnick, P 3-17, pg 82) Electrons incident on a crystal undergo refraction due to an attractive potential of about 15 V that crystals present to electrons (due to the positive ions in the crystal lattice). If the angle of incidence of an electron beam is 45°, and the electrons have an incident energy of 100 eV, what is the angle of refraction? [10]

   Solution: Basically the setup is that this beam of electrons is coming down to strike a surface at 45°, however the surface is attracting them, so they change their direction a bit and end up striking the surface at an angle closer to straight on. The question is asking what angle they strike at. (Actually the question is asking what angle they will be going just AFTER they pass through the surface, which would be the same as the angle they are going just before they pass through the surface.)

   A similar "mechanics" problem might be something like: "A gun is fired towards the ground at an angle of 45°, from a height of 100 meters. At what angle do the bullets hit the ground? The gun fires bullets at 300 meters per second."

   More accurately, the problem we have does not give you distances and speeds, but only energies and electric potentials (which when multiplied by charges are also energies), so maybe a more accurate "translation" might be something like: "A gun is fired towards the ground at an angle of 45°. As the bullets move towards the ground they gain 15 Joules of gravitational potential energy. At what angle do the bullets hit the ground? The gun fires bullets so that they have 100 Joules of kinetic energy."

   Fortunately the energies of the problem are small enought that we do not need to use relativity, classical expressions are sufficient.

   The initial kinetic energy of the electrons is $K_i = 100\,\text{eV}$ and after going through the attractive potential they will have a final kinetic energy of $K_f = 115\,\text{eV}$. Their initial and final speeds are given by:

$$K = \frac{1}{2}mv^2 \quad \Longrightarrow \quad v = \sqrt{\frac{2K}{m}}$$
$$\therefore v_i = \sqrt{\frac{2K_i}{m}} \qquad v_f = \sqrt{\frac{2K_f}{m}}$$

The angle of incidence the electron beam makes with the perpendicular to the crystal plane is $\theta_i = 45°$, so if we call θ_f the final angle of refraction the beam makes with the perpendicular to the crystal plane, a bit of geometry will allow us to break the velocity vectors into components. We will take the x-direction to be parallel to the surface of the plane, and the y-direction to be perpendicular to the surface of the plane, and the angles are measured away from the y axis. The velocity components are thus

$$v_{ix} = v_i \sin\theta_i = \sqrt{\frac{2K_i}{m}}\sin\theta_i, \qquad v_{fx} = v_f \sin\theta_f = \sqrt{\frac{2K_f}{m}}\sin\theta_f,$$
$$v_{iy} = v_i \cos\theta_i = \sqrt{\frac{2K_i}{m}}\cos\theta_i, \qquad v_{fy} = v_f \cos\theta_f = \sqrt{\frac{2K_f}{m}}\cos\theta_f.$$

Since the electric forces are all in the y-direction, the velocity component in the x-direction remains unchanged, so

$$\begin{aligned}
v_{fx} &= v_{ix}, \\
\sqrt{\frac{2K_f}{m}}\sin\theta_f &= \sqrt{\frac{2K_i}{m}}\sin\theta_i, \\
\sqrt{K_f}\sin\theta_f &= \sqrt{K_i}\sin\theta_i, \\
\sin\theta_f &= \sqrt{\frac{K_i}{K_f}}\sin\theta_i, \\
\theta_f &= \sin^{-1}\left(\sqrt{\frac{K_i}{K_f}}\sin\theta_i\right) = \sin^{-1}\left(\sqrt{\frac{100\,\text{eV}}{115\,\text{eV}}}\sin(45°)\right) = \sin^{-1}\left(\sqrt{\frac{(100)}{(115)(2)}}\right), \\
&= \sin^{-1}\left(\sqrt{0.434782\ldots}\right) = \sin^{-1}(0.659380\ldots) = (41.252\ldots)° \\
\theta_f &\approx 41.3°.
\end{aligned}$$

Thus the electrons are refracted at an angle of about 41.3°.

3. (From Eisberg & Resnick, P 3-18, pg 82) What accelerating voltage would be required for electrons in an electron microscope to obtain the same ultimate resolving power as that which could be obtained from a "γ-ray microscope" using 0.2 MeV γ rays? [10]

Solution: In order to have the same resolving power, the electrons need to have the same wavelength as the γ-rays. The γ-ray photons have a wavelength of:

$$\lambda_p = \frac{h}{p_p} = \frac{hc}{E_p} = \lambda_e$$

$$\lambda_e = \frac{h}{p_e} \quad \Longrightarrow \quad p_e = \frac{h}{\lambda_e} = \frac{hE_p}{hc} = \frac{E_p}{c}$$

Since the energies that we are dealing with are similar in magnitude to the rest mass energy of the electron (0.2 vs. 0.5 MeV), we need to use relativistic forms of energy and momentum.

The relativistic energy and momentum relationships and some algebra gives us:

$$\begin{aligned} E_e^2 &= p_e^2 c^2 + m_e^2 c^4 && \text{relativistic energy momentum relationship} \\ &= \frac{E_p^2}{c^2} c^2 + m_e^2 c^4 \\ E_e &= \sqrt{E_p^2 + m_e^2 c^4} \end{aligned}$$

$$\begin{aligned} K_e &= E_e - m_e c^2 && \text{relativistic kinetic energy} \\ &= \sqrt{E_p^2 + m_e^2 c^4} - m_e c^2 \\ &= \sqrt{(0.2\,\mathrm{MeV})^2 + (0.511\,\mathrm{MeV})^2} - (0.511\,\mathrm{MeV}) \\ &= (0.03774\ldots\ \mathrm{MeV}) \end{aligned}$$

Thus to get the electrons to have the appropriate wavelenght, we would need to use an accelerating voltage of about $37.7\,\mathrm{MeV}$.

4. (From problem 2-20, "Simple Nature", Crowell, pg 105) Use the Heisenberg uncertainty principle to estimate the minimum velocity of a proton or neutron in a ^{208}Pb nucleus, which has a diameter of about 13 fm ($1\,\mathrm{fm} = 10^{-15}\,\mathrm{m}$). Assume that the speed is non-relativistic, and then check at the end whether this assumption was warranted. [10]

   Solution: This type of calculation using the Heisenberg uncertainty principle will only give us an approximate answer, so we can ignore the question of whether the uncertainty in position of the nucleon is d or $d/2$, and for simplicity just use $\Delta x = d = 13\,\mathrm{fm}$.

$$\begin{aligned} \Delta x \Delta p &\geq \frac{\hbar}{2} = \frac{h}{4\pi} && \text{Heisenberg uncertainty principle} \\ \Delta p &\geq \frac{\hbar}{2\Delta x} = \frac{\hbar}{2d} \\ m\Delta v &\geq \frac{\hbar}{2d} \\ \Delta v &\geq \frac{\hbar}{2dm} = \frac{h}{4\pi dm} \end{aligned}$$

   The argument here is that since we have an expression for the uncertainty of the nucleon's velocity, the velocity of the nucleon must be *at least* this minimum value, any measurement of the velocity must give a value greater than this value. Since this is all very approximate, we can just use a couple of significant figures in our calculation

$$\begin{aligned} v_{min} &= \frac{\hbar}{2dm} = \frac{h}{4\pi dm} \\ &\approx \frac{(1.05 \times 10^{-34}\,\mathrm{m^2 kg/s})}{2(1.3 \times 10^{-14}\,\mathrm{m})(1.67 \times 10^{-27}\,\mathrm{kg})} \approx 2.4 \times 10^6\,\mathrm{m/s} \end{aligned}$$

   The calculated value of $2.4 \times 10^6\,\mathrm{m/s}$ is about 0.8% of the speed of light, so our non-relativistic assumption was justified.

5. (From Eisberg & Resnick, P 4-8, pg 121, with modifications)

   (a) Show that the fraction of α-particles scattered by an angle Θ or larger in Rutherford scattering is [10]

$$f = \left(\frac{1}{4\pi\epsilon_0}\right)^2 \pi\rho t \left(\frac{zZe^2}{Mv^2}\right)^2 \cot^2(\Theta/2). \tag{1}$$

Solution: Following the derivation in Eisberg & Resnick, we arrive at an expression for $N(\Theta)\mathrm{d}\Theta$, the number of α-particles scattered in the range from Θ to $\Theta + \mathrm{d}\Theta$, namely

$$N(\Theta)\mathrm{d}\Theta = \left(\frac{1}{4\pi\epsilon_0}\right)^2 \left(\frac{zZe^2}{2Mv^2}\right)^2 \frac{I\rho t 2\pi \sin\Theta \mathrm{d}\Theta}{\sin^4(\Theta/2)}. \tag{2}$$

If we divide by I and integrate from Θ_1 to Θ_2 we will get the fraction of the particles scattered between the two angles. If we take $\Theta_2 = \pi$ (the largest scattering angle possible), we will have the desired result:

$$\begin{aligned} f &= \int_{\Theta_1}^{\pi} \frac{N(\Theta)}{I}\mathrm{d}\Theta \\ &= \int_{\Theta_1}^{\pi} \left(\frac{1}{4\pi\epsilon_0}\right)^2 \left(\frac{zZe^2}{2Mv^2}\right)^2 \frac{I\rho t 2\pi \sin\Theta}{I \sin^4(\Theta/2)}\mathrm{d}\Theta \\ &= \left(\frac{1}{4\pi\epsilon_0}\right)^2 \pi\rho t \left(\frac{zZe^2}{Mv^2}\right)^2 \int_{\Theta_1}^{\pi} \frac{\sin\Theta}{2\sin^4(\Theta/2)}\mathrm{d}\Theta. \end{aligned} \tag{3}$$

Comparing (1) and (3) we see that we must show that the integral

$$\int_{\Theta_1}^{\pi} \frac{\sin\Theta}{2\sin^4(\Theta/2)}\mathrm{d}\Theta = \cot^2(\Theta/2).$$

To do this integral we will need to make use of the half angle trig identity that $\sin(2A) = 2\sin(A)\cos(A)$ so $\sin(\Theta) = 2\sin(\Theta/2)\cos(\Theta/2)$, as well as the substitution $u = \sin(\Theta/2)$ which gives us $2\mathrm{d}u = \cos(\Theta/2)\mathrm{d}\Theta$ and do the integral directly. Putting these together gives us

$$\begin{aligned} \int_{\Theta_1}^{\pi} \frac{\sin\Theta}{2\sin^4(\Theta/2)}\mathrm{d}\Theta &= \int_{\Theta_1}^{\pi} \frac{2\sin(\Theta/2)\cos(\Theta/2)}{2\sin^4(\Theta/2)}\mathrm{d}\Theta \\ &= \int_{\Theta_1}^{\pi} \frac{\cos(\Theta/2)}{\sin^3(\Theta/2)}\mathrm{d}\Theta \\ &= \int_{\sin(\Theta_1/2)}^{\sin(\pi/2)} \frac{2}{u^3}\mathrm{d}u \\ &= \left[\frac{-1}{u^2}\right]_{\sin(\Theta_1/2)}^{\sin(\pi/2)} \\ &= \left[\frac{-1}{\sin^2(\pi/2)} - \frac{-1}{\sin^2(\Theta_1/2)}\right] \\ &= \left[-1 + \frac{1}{\sin^2(\Theta_1/2)}\right] \\ &= \left[\frac{-\sin^2(\Theta_1/2) + 1}{\sin^2(\Theta_1/2)}\right] \\ &= \left[\frac{\cos^2(\Theta_1/2)}{\sin^2(\Theta_1/2)}\right] \\ &= \cot^2(\Theta_1/2) \end{aligned}$$

Alternatively, the substitution $x = (\Theta/2)$ which gives us $\Theta = 2x$ and $\mathrm{d}\Theta = 2\mathrm{d}x$, and then looking up the result in in a table gives us

$$\begin{aligned}
\int_{\Theta_1}^{\pi} \frac{\sin\Theta}{2\sin^4(\Theta/2)}\mathrm{d}\Theta &= \int_{\Theta_1}^{\pi} \frac{2\sin(\Theta/2)\cos(\Theta/2)}{2\sin^4(\Theta/2)}\mathrm{d}\Theta \\
&= \int_{\Theta_1}^{\pi} \frac{\cos(\Theta/2)}{\sin(\Theta/2)\sin^2(\Theta/2)}\mathrm{d}\Theta \\
&= \int_{\Theta_1}^{\pi} \cot(\Theta/2)\csc^2(\Theta/2)\mathrm{d}\Theta \\
&= \int_{\Theta_1/2}^{\pi/2} \cot(x)\csc^2(x)2\mathrm{d}x \\
&= 2\int_{\Theta_1/2}^{\pi/2} \cot(x)\csc^2(x)\mathrm{d}x \\
&= 2\left[\frac{-\cot^2(x)}{2}\right]_{\Theta_1/2}^{\pi/2} = \left[-\cot^2(x)\right]_{\Theta_1/2}^{\pi/2} \\
&= -\left[\cot^2(\pi/2) - \cot^2(\Theta_1/2)\right] = -\left[0 - \cot^2(\Theta_1/2)\right] \\
&= \cot^2(\Theta_1/2)
\end{aligned}$$

In either case we get

$$\int_{\Theta_1}^{\pi} \frac{\sin\Theta}{2\sin^4(\Theta/2)}\mathrm{d}\Theta = \cot^2(\Theta_1/2). \tag{4}$$

Putting (4) into (3) gives us our desired result (1) for all angles greater than Θ.

(b) What fraction of 5.59 MeV α-particles from the decay of ^{222}Rn, incident on a gold foil of thickness 1 μm, will be deflected by an angle of $\pi/2$ rad or larger? [5]

Solution: For the gold foil we can get some physical properties from the web site http://www.chemicalelements.com/. The atomic number, and thus the atomic charge of gold is $Z = 79$. The mass density of gold is about 19.32 g/cm^3, and the atomic mass of gold is about 196.96655 amu = 196.96655 g/mol. Since 1 mol = $6.02214199 \times 10^{23}$ atoms, the number of atoms of gold per cubic centimetre is about:

$$\begin{aligned}
\rho &= \frac{19.32\,\mathrm{g}}{\mathrm{cm}^3}\frac{\mathrm{mol}}{196.96655\,\mathrm{g}}\frac{6.02214199\times10^{23}\,\mathrm{atoms}}{\mathrm{mol}} \\
&= 5.90698\ldots\times10^{22}\,\mathrm{atoms/cm}^3 \\
\rho &\approx 5.91\times10^{22}\,\mathrm{atoms/cm}^3 = 5.91\times10^{28}\,\mathrm{atoms/m}^3
\end{aligned}$$

For the α-particles, we only need to know their charge since as seen below, their mass and velocity information come from their given kinetic energy. If we wanted, we could get some physical properties from the web site http://physics.nist.gov/. For a α-particle (a helium nuclei), the mass is about $M = 6.64465598 \times 10^{-27}$ kg and their charge is $z = 2$. With a kinetic energy of 5.59 MeV = $8.95616642 \times 10^{-13}$ J (this is much less than the rest mass energy of about 4000 MeV so we can use non-relativistic forms of energy and momentum) we can find the particle's speed by

$$K = \frac{1}{2}Mv^2 \quad\Longrightarrow\quad Mv^2 = 2K.$$

With these values, (1) then gives us

$$
\begin{aligned}
f &= \left(\frac{1}{4\pi\epsilon_0}\right)^2 \pi\rho t \left(\frac{zZe^2}{2K}\right)^2 \cot^2(\Theta/2) \\
&= (8.9876 \times 10^9\,\mathrm{Nm^2/C^2})^2(\pi)(5.91 \times 10^{28}\,\mathrm{atoms/m^3})(10^{-6}\,\mathrm{m}) \left(\frac{zZe^2}{2K}\right)^2 \cot^2(\Theta/2) \\
&= (1.499\ldots \times 10^{43}\,\mathrm{atoms\,N^2m^2/C^4}) \left(\frac{zZe^2}{2K}\right)^2 \cot^2(\Theta/2) \\
&= (1.499\ldots \times 10^{43}\,\mathrm{atoms\,N^2m^2/C^4}) \left(\frac{(2)(79)(1.6022 \times 10^{-19}\,\mathrm{C})^2}{(2)(8.95616642 \times 10^{-13}\,\mathrm{J})}\right)^2 \cot^2(\pi/4) \\
&= (1.499\ldots \times 10^{43}\,\mathrm{atoms\,N^2m^2/C^4})(5.127\ldots \times 10^{-48}\,\mathrm{C^4/N^2m^2})(1) \\
&= 7.6895\ldots \times 10^{-5} \\
&\approx 7.69 \times 10^{-5}.
\end{aligned}
$$

A fraction of about (7.69×10^{-5}) or one out of 13,000 of the α-particles will be deflected by more than 90°.

Headstart for next week, Week 06, starting Monday 2004/10/18:
– Read Chapter 2.4 “The Atom” in “Simple Nature” by Crowellk
– Read Chapter 4 “Bohr’s Model of the Atom” in Eisberg & Resnick
– – Section 4.5 “Bohr’s Postulates”
– – Section 4.6 “Bohr’s Model”
– – Section 4.7 “Correction for Finite Nuclear mass”
– – Section 4.8 “Atomic Energy States”
– – Section 4.9 “Interpretation of the Quantization Rules”
– – Section 4.10 “Sommerfeld’s Model”
– – Section 4.11 “The Correspondence Principle”
– – Section 4.12 “A Critique of the Old Quantum Theory”