Physics 202H - Introductory Quantum Physics I
Homework #04 - Solutions

Fall 2004 — Due 5:01 PM, Tuesday 2004/10/12

[70 points total]

"Journal" questions. Briefly share your thoughts on the following questions:
– Give an example of a time you made use of physics knowledge you gained from a physics course, outside of schoolwork. What physics phenomena have you noticed outside of the classroom? Have you noticed or made use of *quantum physics* outside of class? In what context?
– Any comments about this week's activities? Course content? Assignment? Lab?

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1. (From Eisberg & Resnick, Q 2-23, pg 52) What is wrong with taking the geometrical interpretation of a cross section as literally true? Limit your discussion to about 50 words or so. [10]

   Solution: The geometrical interpretation implies that the target particles take up a certain fraction of the target's area, and any particles that hit in that area will be affected. In fact, each particle that strikes the target at any location on the target has a *chance* of being affected, depending on what the cross section is. For a large number of small particles, the differences between the geometrical and the probabilistic interpretations in terms of calculations are practically non-existent, but they would lead to very different answers if applied at the microscopic level.

2. (From Eisberg & Resnick, P 2-29, pg 54) A particular pair is produced such that the positron is at rest and the electron has a kinetic energy of 1.0 MeV moving in the direction of flight of the pair-producing photon.

   (a) Neglecting the energy transferred to the nucleus of the nearby atom, what is the energy of the incident photon? [5]

   Solution: The final energy is made up of the kinetic energy of the electron plus the mass energy of the electron and the positron. If we neglect the energy transferred to the nucleus, the initial energy of the incident photon should be equal to this final energy:

$$\begin{aligned} E_{photon} = E_i = E_f &= K + m_{e^+}c^2 + m_{e^-}c^2 \\ &= K + 2m_e c^2 \\ &= 1.0\,\text{MeV} + 2(0.5110\,\text{MeV}) \\ &= 2.022\,\text{MeV} \\ E_{photon} &\approx 2.0\,\text{MeV} \approx 3.2 \times 10^{-13}\,\text{J} \end{aligned}$$

(b) What percentage of the photon's momentum is transferred to the nucleus? [5]

Solution: The momentum of the photon can be calculated from its energy, and the momentum of the electron (the only thing moving at the end of the interaction) can also be calculated, so the nucleus will have momentum equal to the difference between the two. We need to establish some relationships between kinetic energy, total energy, and momentum for the electron, namely:

$$K = E - m_0c^2 \quad \Longrightarrow \quad E = K + m_0c^2 \quad \Longrightarrow \quad E^2 = K^2 + 2Km_0c^2 + m_0^2c^4$$

$$\begin{aligned} E^2 = p^2c^2 + m_0^2c^4 \quad \Longrightarrow \quad p^2c^2 &= E^2 - m_0^2c^4 \\ &= K^2 + 2Km_0c^2 + m_0^2c^4 - m_0^2c^4 \\ &= K^2 + 2Km_0c^2 \\ pc &= \sqrt{K^2 + 2Km_0c^2} \end{aligned}$$

Thus for the electron, p_{e^-} is given by

$$p_{e^-} = \frac{\sqrt{K^2 + 2Km_ec^2}}{c}.$$

The momentum of the nucleus is then

$$\begin{aligned} p_{nucleus} &= p_{photon} - p_{e^-} \\ &= \frac{h}{\lambda_{photon}} - \frac{\sqrt{K^2 + 2Km_ec^2}}{c} \\ &= \frac{E_{photon}}{c} - \frac{\sqrt{K^2 + 2Km_ec^2}}{c} \\ &= \frac{K + 2m_ec^2}{c} - \frac{\sqrt{K^2 + 2Km_ec^2}}{c} \\ p_{nucleus}c &= K + 2m_ec^2 - \sqrt{K^2 + 2Km_ec^2} \end{aligned}$$

Since we want the percentage, we need to divide through by $p_{photon}c$ and multiply by 100%:

$$\begin{aligned} \frac{p_{nucleus}}{p_{photon}} &= \frac{K + 2m_ec^2 - \sqrt{K^2 + 2Km_ec^2}}{K + 2m_ec^2} \\ &= 1 - \frac{\sqrt{K^2 + 2Km_ec^2}}{K + 2m_ec^2} \\ &= 1 - \frac{(1.4219\ldots\ \text{MeV})}{(2.022\,\text{MeV})} \\ &= 1 - (0.7032\ldots) \\ &= (0.2967\ldots) \end{aligned}$$

Thus approximately 29.7% of the photon's momentum is transferred to the nucleus.

3. (From Eisberg & Resnick, P 2-34, pg 54) What is the thickness of a lead slab which will attenuate a beam of 10 keV x rays by a factor of 100? Use the data of Eisberg & Resnick, Figure 2-17, pg 49. [10]

Solution: The word "attenuate" means "to reduce in volume", so we want to find the thickness of lead that will reduce the intensity of the beam by a factor of 100, ie. so that the initial intensity is 100 times greater than the final intensity: $I_i = 100 I_f$. Using the data from the table in the text, we find that the total cross section for lead interacting with 10 keV x rays is somewhere between 10^{-20} and $10^{-19}\,\mathrm{cm}^2$. In order to estimate this value of σ, we need to realized that the vertical scale of the graph is logarithmical rather than linear, and σ is NOT around $1.4 \times 10^{-20}\,\mathrm{cm}^2$ but rather $\sigma \approx 10^{-19.6}\,\mathrm{cm}^2 \approx 2.5 \times 10^{-20}\,\mathrm{cm}^2$.

For lead, we can get some physical properties from http://www.chemicalelements.com/. The mass density of lead is about $11.34\,\mathrm{g/cm^3}$, and the atomic mass of lead is about $207.2\,\mathrm{amu} = 207.2\,\mathrm{g/mol}$. Since $1\,\mathrm{mol} = 6.02214199 \times 10^{23}$ atoms, the number of atoms of lead per cubic centimetre is about:

$$\begin{aligned}\rho &= \frac{11.34\,\mathrm{g}}{\mathrm{cm}^3}\frac{\mathrm{mol}}{207.2\,\mathrm{g}}\frac{6.02214199 \times 10^{23}\,\mathrm{atoms}}{\mathrm{mol}} \\ &= 3.29590\ldots \times 10^{22}\,\mathrm{atoms/cm^3} \approx 3.3 \times 10^{22}\,\mathrm{atoms/cm^3}\end{aligned}$$

Note that the units of this parameter are atoms/cm^3, but in most cases we will ignore the "atoms" part to the same extent that we ignore the units of "chance of interaction per atom" part of the units for cross sections.

Given these values, since we know how the intensity is a function of the material's thickness, we can compute the necessary thickness:

$$\begin{aligned}I_f &= I_i \mathrm{e}^{-\sigma\rho t} \quad \Longrightarrow \quad \frac{I_f}{I_i} = \mathrm{e}^{-\sigma\rho t} \\ \frac{I_i}{I_f} &= \mathrm{e}^{\sigma\rho t} \quad \Longrightarrow \quad \ln\left(\frac{I_i}{I_f}\right) = \sigma\rho t \\ t &= \frac{1}{\sigma\rho}\ln\left(\frac{I_i}{I_f}\right) \\ &= \frac{1}{(2.5 \times 10^{-20}\,\mathrm{cm}^2)(3.3 \times 10^{22}\,1/\mathrm{cm}^3)}\ln(100) \\ &= 5.562\ldots \times 10^{-3}\,\mathrm{cm} \\ &\approx 56\,\mu\mathrm{m} = 5.6 \times 10^{-7}\,\mathrm{m}\end{aligned}$$

4. (From Eisberg & Resnick, Q 3-11, pg 80) Could crystallographic studies be carried out with protons? With neutrons? Limit your discussion to about 50 words or so. [10]

   Solution: Crystallographic studies rely on the interference effects of the incident beam when reflected or refracted off of the various planes making up the crystal lattice being studied. This type of interference depends on the wavelength of the beam, and the interaction between the beam constituents and the atoms that make up the crystal. Just like photons, protons and neutrons do have wavelengths, so they can all be used for crystallographic studies.

   Using electrons or protons for crystallography is certainly possible, but since protons are much more massive than electrons, they will have much larger wavelengths for the same energy/accelerating potential. Equivalently the massive particle will need to be moving slower than the less massive particle to have the same wavelength. Neutrons would have similar wavelengths to protons of the same energy (since they have virtually the same mass), but the interaction between the neutrons and the crystal would have to be a nuclear interaction rather than an electromagnetic interaction since the neutron is electrically uncharged. Thus the neutron would be more sensitive to the nuclear structure of the crystal constituents than the charged particles which would be more sensitive to the molecular structures, influenced of course by the overall energies of the probing particles.

   Web searches turn up a wealth of information about "neutron crystallography" and "electron crystallography", but essentially nothing for "proton crystallography", reflecting the fact that using protons does not give one significant advantages over using electrons, and electrons are much much easier to work with.

5. (From Eisberg & Resnick, P 3-2, pg 81) The wavelength of the yellow spectral emission of sodium is $\lambda = 5890\,\text{Å}$. At what kinetic energy would an electron have the same de Broglie wavelength? [10]

   Solution: We know that the de Broglie wavelength is given by $\lambda = h/p$, so the momentum of the electron with this wavelength is given by $p = h/\lambda$. For the non-relativistic case, we know that $p = mv$ and $K = mv^2/2$ so

$$\begin{aligned} K &= \frac{1}{2}mv^2 = \frac{p^2}{2m} = \frac{h^2}{2m\lambda^2} \\ &= \frac{(4.1356668\ldots \times 10^{-15}\,\text{eV}\cdot\ \text{s})^2}{2(0.511 \times 10^6\,\text{eV}/c^2)(5.89 \times 10^{-7}\,\text{m})^2} \\ &= 4.3356\ldots \times 10^{-6}\,\text{eV} \approx 4.34\,\mu\text{eV} \approx 6.95 \times 10^{-25}\,\text{J} \end{aligned}$$

   Since this is such a small energy compared to the rest mass energy of the electron of 0.511 MeV, the non-relativistic result is valid.

6. (From Eisberg & Resnick, P 3-7, pg 81) A particle of charge e and rest mass m_0 is accelerated to relativistic speeds by an accelerating potential V.

Solution: From this information, we know that the particle has kinetic energy of $K = eV$.

(a) Show that the de Broglie wavelength of the particle is given by: [10]

$$\lambda = \frac{h}{\sqrt{2m_0 eV}} \left(1 + \frac{eV}{2m_0c^2}\right)^{-\frac{1}{2}}$$

Solution: Since this problem is explicitly relativistic, we cannot use the non-relativistic expressions relating particle momentum and kinetic energy. Instead we must use the proper relativistic ones, namely:

$$K = E - m_0c^2 \quad \Longrightarrow \quad E = K + m_0c^2 = eV + m_0c^2 \quad \Longrightarrow \quad E^2 = e^2V^2 + 2eVm_0c^2 + m_0^2c^4$$

$$E^2 = p^2c^2 + m_0^2c^4 \quad \Longrightarrow \quad p^2c^2 = E^2 - m_0^2c^4$$

With the de Broglie relationship $\lambda = h/p$ we have:

$$\begin{aligned} p^2c^2 &= E^2 - m_0^2c^4 \\ &= e^2V^2 + 2eVm_0c^2 + m_0^2c^4 - m_0^2c^4 \\ &= e^2V^2 + 2eVm_0c^2 \\ \frac{1}{p^2c^2} &= \frac{1}{e^2V^2 + 2eVm_0c^2} = \left(\frac{1}{eV}\right)\left(\frac{1}{eV + 2m_0c^2}\right) \qquad (1) \\ \lambda^2 = \frac{h^2}{p^2} &= \left(\frac{h^2c^2}{eV}\right)\left(\frac{1}{eV + 2m_0c^2}\right). \end{aligned}$$

Unfortunately it is unclear how to get from this stage to the given relationship. However, if we work from the given relationship backwards, we should be able to arrive at the point where we have gotten to thus far.

$$\begin{aligned} \lambda = \frac{h}{p} &= \frac{h}{\sqrt{2m_0eV}}\left(1 + \frac{eV}{2m_0c^2}\right)^{-\frac{1}{2}} \qquad (2) \\ \frac{h^2}{p^2} &= \frac{h^2}{2m_0eV}\left(1 + \frac{eV}{2m_0c^2}\right)^{-1} \\ &= \frac{h^2}{2m_0eV}\left(\frac{2m_0c^2 + eV}{2m_0c^2}\right)^{-1} \\ &= \frac{h^2}{2m_0eV}\left(\frac{2m_0c^2}{2m_0c^2 + eV}\right) \\ &= \frac{h^2c^2}{eV\left(2m_0c^2 + eV\right)} \\ \frac{1}{p^2} &= \frac{c^2}{eV\left(2m_0c^2 + eV\right)} \\ \frac{1}{p^2c^2} &= \frac{1}{eV\left(2m_0c^2 + eV\right)} = \left(\frac{1}{eV}\right)\left(\frac{1}{eV + 2m_0c^2}\right). \qquad (3) \end{aligned}$$

Since (1) is identical to (3), we have shown that the relation (2) holds.

(b) Show how this agrees with $\lambda = h/p$ in the non-relativistic limit. [10]

Solution: It should be noted that $\lambda = h/p$ is always true, not only in the non-relativistic limit (we used it in the first part of this question), and the question is actually asking us to demonstrate that when using a non-relativistic approximations for p, K, etc. that (2) can be shown to be equivalent to $\lambda = h/p = h/mv$.

In the non-relativistic limit, (2) can be simplified since the kinetic energy $K = eV$ is much smaller than the rest-mass energy m_0c^2, thus $eV/2m_0c^2 \approx 0$ and $K \approx p^2/2m_0$ or $p^2 \approx 2m_0K = 2m_0eV$. Thus (2) becomes

$$\lambda = \frac{h}{\sqrt{2m_0eV}}\left(1 + \frac{eV}{2m_0c^2}\right)^{-\frac{1}{2}} \approx \frac{h}{\sqrt{p^2}}\,(1+0)^{-\frac{1}{2}} = \frac{h}{p}$$

Headstart for next week, Week 05, starting Tuesday 2004/10/12:
– Read Chapter 2.3 “Matter as a Wave” in “Simple Nature” by Crowellk
– Read Chapter 3 “De Broglie’s Postulate – Wavelike Properties of Particles” in Eisberg & Resnick
– – Section 3.3 “The Uncertainty Principle”
– – Section 3.4 “Properties of Matter Waves”
– – Section 3.5 “Some Consequences of the Uncertainty Principle”
– – Section 3.6 “The Philosophy of Quantum Theory”
– Read Chapter 2.4 “The Atom” in “Simple Nature” by Crowellk
– Read Chapter 4 “Bohr’s Model of the Atom” in Eisberg & Resnick
– – Section 4.1 “Thompson’s Model”
– – Section 4.2 “Rutherford’s Model”
– – Section 4.3 “The Stability of the Nuclear Atom”
– – Section 4.4 “Atomic Spectra”