Physics 202H - Introductory Quantum Physics I
Homework #01 - Solutions

Fall 2004 Due 5:01 PM, Monday 2004/09/20

[70 points total]

"Journal" questions. Briefly share your thoughts on the following questions:
– What are your goals for the course? What are your expectations for the course?
– Any comments about this week's activities? Course content? Assignment? Lab?

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1. Electronic communications:
   (a) Send me (jbeda@trentu.ca) an e-mail message from your trentu.ca account, with a subject of "202H–HW–01" [5]
   Solution: Send the email message.
   (b) Sign onto WebCT and post a message in the discussion forum "General social 'discussions". [5]
   Solution: Post the WebCT message.
   (c) Put your name and email address and phone number inside your texts and on your calculator and anything else you might misplace - it will not prevent theft, but it will allow anyone who finds your stuff to have a chance of returning it. [0]
   Solution: Break out the pen/pencil and inscribe.

2. A relativistic car:
   (a) How fast must a car of length L be traveling in order to fit into a garage of length $L/2$, ie. in the garage rest frame at what speed is the car's length equal to the proper length of the garage? [5]
   Solution: In the car's rest frame (traveling along with the car), the car is of length L. In the garage rest frame, the car is shorter by a factor of γ, or $L' = L/\gamma$. We are given that we want the contracted length to be $L' = L/2$ thus we have

$$L' = \frac{L}{\gamma} = \frac{L}{2} \quad \Longrightarrow \quad \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = 2.$$

   From this we get:

$$\begin{aligned}
\sqrt{1 - \frac{v^2}{c^2}} &= \frac{1}{2} \\
1 - \frac{v^2}{c^2} &= \frac{1}{4} \\
1 - \frac{1}{4} &= \frac{v^2}{c^2} \\
\frac{3}{4} &= \frac{v^2}{c^2} \\
\frac{3}{4} &= \frac{v^2}{c^2} \\
\frac{v}{c} = \frac{\sqrt{3}}{2} \quad &\Longrightarrow \quad v = c\frac{\sqrt{3}}{2}.
\end{aligned}$$

   Thus the car must be traveling at a speed of $c\frac{\sqrt{3}}{2} \approx (0.8660)c \approx (2.5962 \times 10^8)$ m/s in order for its length L' as measured in the garage reference frame to be $\frac{L}{2}$.

(b) If a car of length L is traveling at speed $c/2$, how long does it take for the car to travel past an observer, in the observer's rest frame? How long does it take for the car to travel past an observer in the car's rest frame? [5]

Solution: In this case the car is traveling at a given speed past the observer, so the length in the observer's rest frame is shorter by a factor of γ, namely $L' = L/\gamma$. The car is traveling at a speed $v = c/2$ so the time to go length L' is given by:

$$\Delta t' = \frac{\Delta x'}{v} = \frac{L'}{v} = \frac{L2}{\gamma c}$$

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1}{\sqrt{1-\frac{c^2}{4c^2}}} = \frac{1}{\sqrt{1-\frac{1}{4}}} = \frac{1}{\sqrt{\frac{3}{4}}} = \frac{2}{\sqrt{3}}$$

$$\Delta t' = \frac{L2\sqrt{3}}{2c} = \sqrt{3}\frac{L}{c}.$$

So in the rest frame of the observer, it takes a time of $\sqrt{3}\frac{L}{c} \approx (1.732)\frac{L}{c}$ for the car to pass.

From the point of view of the car, however, the length is unchanged at L and the speed is still given by $v = c/2$ so the time needed to pass the observer's position is given by:

$$\Delta t = \frac{\Delta x}{v} = \frac{L}{v} = \frac{L2}{c} = 2\frac{L}{c}.$$

In the car's rest frame, the observer goes past in a time of $2\frac{L}{c}$. Since $2 > \sqrt{3}$, the "car time" is greater than the "observer time", which seems to be at odds with the idea that "moving clocks tick slower" as is often stated, and in contradiction with problems where we send a spaceship off on a long fast journey. There is no actual contradiction, since this problem is different from the spaceship journey type of problem in that they each measure different distances, and thus different times too.

3. (From problem "Simple Nature", Crowell, 1-9, pg 39) A free neutron (as opposed to a neutron bound into an atomic nucleus) is unstable, and decays radioactively into a proton, an electron, and a particle called a neutrino. (This process can also occur for a neutron in a nucleus, but then other forms of mass-energy are involved as well.) The masses are as follows:

neutron	$m_n = 1.67495 \times 10^{-27}$ kg
proton	$m_p = 1.67265 \times 10^{-27}$ kg
electron	$m_e = 0.00091 \times 10^{-27}$ kg
neutrino	$m_\nu \approx 0$ kg, negligible mass

(a) Find the energy released in the decay of a free neutron. [5]

Solution: The initial energy is just the rest mass energy of the free neutron, $E_i = m_n c^2$. The final energy of the system is the rest mass energies of the resulting decay products plus the energy released as kinetic energy K, thus $E_f = m_p c^2 + m_e c^2 + m_\nu c^2 + K$. Energy conservation gives us:

$$E_i = E_F$$

$$m_n c^2 = m_p c^2 + m_e c^2 + m_\nu c^2 + K$$

$$K = m_n c^2 - m_p c^2 - m_e c^2 - m_\nu c^2 = (m_n - m_p - m_e - m_\nu)\, c^2$$

$$K = (1.67495 - 1.67265 - 0.00091 - 0)\, 10^{-27}\text{kg} \times \left(2.99792458 \times 10^8 \text{m/s}\right)^2$$

$$K = \left(0.00139 \times 10^{-27}\text{kg}\right)\left(2.99792458 \times 10^8 \text{m/s}\right)^2$$

$$K = 1.249269698 \times 10^{-13} \text{kg m}^2/\text{s}^2$$

$$K \approx 1.25 \times 10^{-13}\text{J} \approx 7.80 \times 10^5 \text{eV}.$$

The amount of energy released in the decay of a free neutron is about 1.25×10^{-13}J or about 7.80×10^5eV.

(b) We might imagine that a proton could decay into a neutron, a positron, and a neutrino. Although such a process can occur within a nucleus, explain why it cannot happen to a free proton. (If it could, hydrogen would be radioactive!) [5]

Solution: A free proton has a smaller rest mass than a free neutron $m_p < m_n$, thus to decay into a neutron (plus a positron and a neutrino) there would need to be some other energy beyond the proton's rest mass energy to be used. For a free proton, one might think that the proton's kinetic energy could be used, however in the rest frame of the proton, it would have no kinetic energy. Thus in the proton's rest frame, the principle of conservation of mass/energy forbids the proton from decaying into a neutron and other particles. Since the laws of physics are the same in any inertial frame, the decay of a free proton into a neutron and other particles does not occur no matter how fast the proton might be moving.

Equivalently one can use the conservation of momentum to show that regardless of the momentum and kinetic energy of the initial proton, there is no way of having the decay products have the same total momentum and (kinetic + rest mass) energy.

4. (From problem 1-17, "Simple Nature", Crowell, pg 40) Our sun lies at a distance of 26,000 light years from the center of the galaxy, where there are some spectactular sights to see, including a supermassive black hole that is rapidly eating up the surrounding interstellar gas and dust. Rich tourist Bill Gates IV buys a spaceship, and heads for the galactic core at a speed of 99.99999% of the speed of light.

(a) According to observers on Earth, how long does it take before he gets back? (Ignore the short time he actually spends sightseeing at the core.) [5]

Solution: It will take a time of $\Delta t = \Delta x / v$ to make the one way trip, from the point of view of people on Earth. With $\Delta x = 26000\,\text{ly}$ and Bill's speed of $v = 0.9999999c$ where c has a value of one light year per year, this comes to a time interval of just a bit more than 26,000 years:

$$\Delta t = \frac{\Delta x}{v} = \frac{26000\,\text{ly}}{0.9999999\,\text{ly/y}} = \frac{26000}{0.9999999}\,\text{y} \approx 26000.0026\,\text{y}$$

If your calculator cannot figure this out, you could use $0.9999999 = 1 - 10^{-7}$ so that we can use the Taylor series expansion of $(1-x)^{-1}$ for small x.

$$(1-x)^{-1} = \frac{1}{1-x} = \sum_{i=0}^{\infty} x^i = 1 + x + x^2 + x^3 + \cdots$$

With this formula we can just set $x = 10^{-7}$ and basically ignore the terms x^2 and higher since they are so very small.

$$\frac{1}{0.9999999} = \frac{1}{1-10^{-7}} = \left(1 - 10^{-7}\right)^{-1} \approx \left(1 + 10^{-7}\right)$$

$$\Delta t = \frac{26000}{1-10^{-7}}\,\text{y} \approx 26000\left(1+10^{-7}\right)\,\text{y} = (26000 + 0.0026)\,\text{y} = 26000.0026\,\text{y}.$$

To it take Bill about 22 hours and 47.5 minutes more than 26,000 years to make the trip one way, according to people on Earth. For the total journey, we just have to double everything:

$$2\Delta t \approx 52000.0052\,\text{y},$$

about 1 day, 21 hours and 35 minutes more than 52,000 years.

(b) In Bill's frame of reference, how much time passes? [5]

Solution: Bill also knows that he is traveling at $v = 0.9999999c$, but the effects of length contraction mean that he only needs to go a short distance, thus taking a short time:

$$\Delta t' = \frac{\Delta x'}{v} \quad , \quad \Delta x' = \frac{\Delta x}{\gamma} \quad , \quad \gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}}$$

$$\gamma = \frac{1}{\sqrt{1-(0.9999999)^2}} = \frac{1}{\sqrt{1-(0.9999998)}} = \frac{1}{\sqrt{2\times10^{-7}}} \approx \frac{1}{4.472\times10^{-4}} \approx 2236$$

$$\Delta x' = \frac{\Delta x}{\gamma} = 26000\,\text{ly}\sqrt{2\times10^{-7}} = 11.62755348299891\,\text{ly}$$

$$\Delta t' = \frac{\Delta x'}{v} = \frac{11.627553482\,\text{ly}}{0.9999999\,\text{ly/y}} = \frac{11.627553482}{0.9999999}\,\text{y} \approx 11.627554645\,\text{y}$$

As above, the trip takes about one part in 10^7 longer than a beam of light, which comes out to about 37 seconds more. For the round trip, of course, everything is doubled:

$$2\Delta t' \approx 23.25510929\,\text{y},$$

Note that one could also calculate $\Delta t'$ by:

$$\Delta t' = \frac{\Delta t}{\gamma}.$$

Note also that we are using many more significant figures than is justified by the accuracy of the data given (ie. 23,000 ly), but since we are considering travel at only one part in 10^7 slower than c, it is necessary to carry lots of figures in our calculations to get any results different from travel at c.

(c) When you compare your answer to part b with the round-trip distance, do you conclude that Bill considers himself to be moving faster than the speed of light? If so, how do you reconcile this with relativity? If not, then resolve the apparent paradox. [5]

Solution: It only takes Bill about 23 years to travel 52,000 light years, which seems to indicate faster than light speed travel. However, from Bill's point of view, his trip only covered about 23 light years of distance, so there was no conflict with the postulates of relativity. The apparent contradiction arises from making a calculation using the time duration in one frame of reference and the distance in a different frame. Such calculations are not valid and would not be expected to give reasonable results. In Bill's reference frame, light still travels at c in all directions.

5. (From problem A-13, Eisberg & Resnick, pg A-19)

(a) Show that when $v/c < 1/10$, then [15]

i. K/m_0c^2 is less than about 1/200, and

Solution: Note that for all problems of this nature (when we are asked to show that when some parameter is greater (or lesser) than some value that it implies that some result is greater (or lesser) than some other value) we cannot just show that the relationship holds for one specific value, but we need to show that for all values greater (or lesser) than the given value the relationship holds. One cannot just plug in the numerical values and then say "It's done!" At the very least we need to show that the calculated value is an upper (or lower) bound for the value of the result.

The kinetic energy K is related to the rest mass energy so that:

$$\frac{K}{m_0c^2} = \frac{\gamma m_0c^2 - m_0c^2}{m_0c^2} = \frac{m_0c^2(\gamma - 1)}{m_0c^2} = \gamma - 1$$

Given that $v/c < 1/10$ we have the following. Note that γ is always greater than one and as $|v|$ increases, so does γ and as $|v|$ decreases, so does γ, which can be seen by noticing that the derivative of γ with respect to v has a positive value.

$$\gamma - 1 = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} - 1 < \frac{1}{\sqrt{1-\frac{1}{100}}} - 1 = \frac{1}{\sqrt{0.99}} - 1 = 1.005037\ldots - 1 = 0.005037\ldots \approx 0.005 = \frac{1}{200}.$$

Thus (ignoring the extra figures) we have:

$$\frac{K}{m_0c^2} < 0.005037\ldots \approx \frac{1}{200}.$$

In reality we have shown that

$$\frac{K}{m_0c^2} < 0.005037\ldots = \frac{1.007\ldots}{200},$$

but $1/200$ is a bit easier to write. Going the other way if $K/m_0c^2 < 1/200$ then we have that

$$\gamma - 1 = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} - 1 < \frac{1}{200}$$

$$\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} < \frac{1}{200} + 1 = 1.005$$

$$\frac{1}{1-\frac{v^2}{c^2}} < (1.005)^2 = 1.010025$$

$$\frac{1}{1.010025} < 1 - \frac{v^2}{c^2}$$

$$\frac{v^2}{c^2} < 1 - \frac{1}{1.010025}$$

$$\frac{v}{c} < \sqrt{1 - \frac{1}{1.010025}} = 0.0996\ldots,$$

so to get the factor of 200 we need v/c to be a little bit less than $1/10$.
If we use the classical relationships, since they are good to less than 1%, it is fairly easy to show the $1/200$ value without needing to ignore some extra decimal places:

$$\frac{K_c}{m_0c^2} = \frac{\frac{1}{2}m_0v^2}{m_0c^2} = \frac{v^2}{2c^2}$$

Thus:

$$\frac{v}{c} < \frac{1}{10} \implies \frac{K_c}{m_0c^2} = \frac{v^2}{2c^2} < \frac{1}{200}$$

ii. the classical expressions for kinetic energy, $K_c = m_0 v^2/2$, may be used with an error of less than 1%, and

Solution: The classical expressions for kinetic energy and momentum (K_c and p_c) can be compared to the relativistic expressions. The % errors are given by:

$$\left|\frac{K_c - K}{K}\right| \times 100\% \quad \text{and} \quad \left|\frac{p_c - p}{p}\right| \times 100\%$$

$$\begin{aligned} K_c &= \tfrac{1}{2} m_0 v^2, & p_c &= m_0 v, \\ K &= m_0 c^2 (\gamma - 1), & p &= \gamma m_0 v. \end{aligned}$$

For small values of v/c, γ can be approximated by a Taylor expansion in terms of v^2/c^2:

$$(1-x)^{-\frac{1}{2}} = 1 + \frac{1}{2}x + \frac{1\cdot 3}{2\cdot 4}x^2 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^3 + \cdots$$

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} = 1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{1\cdot 3}{2\cdot 4}\frac{v^4}{c^4} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{v^6}{c^6} + \cdots$$

$$(\gamma - 1) = \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \frac{15}{48}\frac{v^6}{c^6} + \cdots$$

With those preliminaries out of the way we can look at the % errors:

$$\left|\frac{K_c - K}{K}\right| = \left|\frac{K_c}{K} - 1\right| = \left|\frac{\frac{1}{2}m_0 v^2}{m_0 c^2(\gamma-1)} - 1\right| = \left|\frac{v^2}{2c^2(\gamma-1)} - 1\right| = \left|\frac{\frac{v^2}{c^2}}{2(\gamma-1)} - 1\right|$$

$$\left|\frac{\frac{v^2}{c^2}}{2(\gamma-1)} - 1\right| = \left|\frac{\frac{v^2}{c^2}}{\frac{v^2}{c^2} + \frac{3}{4}\frac{v^4}{c^4} + \frac{15}{24}\frac{v^6}{c^6} + \cdots} - 1\right| = \left|\frac{1}{1 + \frac{3}{4}\frac{v^2}{c^2} + \frac{15}{24}\frac{v^3}{c^3} + \cdots} - 1\right|$$

To get rid of the pesky absolute value symbols, we note that the fractional part is less than one since the series in the denominator is larger than one:

$$\left|\frac{K_c - K}{K}\right| = 1 - \frac{1}{1 + \frac{3}{4}\frac{v^2}{c^2} + \frac{15}{24}\frac{v^3}{c^3} + \cdots}$$

Ignoring anything but the leading term, and then taking the approximation that $(1+x)^{-1} \approx (1-x)$ we have:

$$\left|\frac{K_c - K}{K}\right| \approx 1 - \frac{1}{1 + \frac{3}{4}\frac{v^2}{c^2}} \approx 1 - \left(1 - \frac{3}{4}\frac{v^2}{c^2}\right) = \frac{3}{4}\frac{v^2}{c^2} < \frac{3}{400} < 1\%$$

Alternatively one can do the % error calculation with the classical term in the denominator (this isn't strictly correct, but for small % error it does not make much difference):

$$\left|\frac{K_c - K}{K_c}\right| = \left|\frac{K}{K_c} - 1\right| = \left|\frac{m_0 c^2(\gamma-1)}{\frac{1}{2}m_0 v^2} - 1\right| = \left|\frac{2c^2(\gamma-1)}{v^2} - 1\right| = \left|\frac{c^2}{v^2}2(\gamma-1) - 1\right|$$

$$\left|\frac{K_c - K}{K_c}\right| = \left|\frac{c^2}{v^2}2(\gamma-1) - 1\right| = \left|\frac{c^2}{v^2}\left(\frac{v^2}{c^2} + \frac{3}{4}\frac{v^4}{c^4} + \frac{15}{24}\frac{v^6}{c^6} + \cdots\right) - 1\right|$$

$$\left|\frac{K_c - K}{K_c}\right| = \left|\left(1 + \frac{3}{4}\frac{v^2}{c^2} + \frac{15}{24}\frac{v^3}{c^3} + \cdots\right) - 1\right|$$

$$\left|\frac{K_c - K}{K_c}\right| = \left|\frac{3}{4}\frac{v^2}{c^2} + \frac{15}{24}\frac{v^3}{c^3} + \frac{105}{192}\frac{v^4}{c^4} + \cdots\right| = \left(\frac{3}{4}\frac{v^2}{c^2}\right)\left|1 + \frac{5}{6}\frac{v}{c} + \frac{35}{48}\frac{v^2}{c^2} + \cdots\right|$$

Since v/c is small, we can ignore all but the leading term and we get:

$$\left|\frac{K_c - K}{K_c}\right| \approx \frac{3}{4}\frac{v^2}{c^2} < \frac{3}{400} < 1\%$$

Alternatively, given the v/c relationship, we can proceed without needing the Taylor expansion:

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} < \frac{1}{\sqrt{1-\frac{1}{100}}} = \frac{1}{\sqrt{\frac{99}{100}}} = \frac{10}{\sqrt{99}} = \frac{10}{3\sqrt{11}} = 1.005037\ldots$$

$$\gamma - 1 < \frac{10}{3\sqrt{11}} - 1 = \frac{10 - 3\sqrt{11}}{3\sqrt{11}} = 005037\ldots$$

$$\left|\frac{K_c - K}{K}\right| = 1 - \frac{\frac{v^2}{c^2}}{2(\gamma - 1)} < 1 - \frac{3\sqrt{11}}{2000 - 600\sqrt{11}} = \frac{2000 - 600\sqrt{11} - 3\sqrt{11}}{2000 - 600\sqrt{11}}$$

$$\left|\frac{K_c - K}{K}\right| < \frac{2000 - 603\sqrt{11}}{2000 - 600\sqrt{11}} = 0.007506\ldots < 1\%$$

iii. the classical expressions for momentum, $p_c = m_0 v$, may be used with an error of less than 1%. **Solution:** We want to do the same sort of thing with the momentum relationships that we did above to the kinetic energy ralationships:

$$\left|\frac{p_c - p}{p}\right| = \left|\frac{p_c}{p} - 1\right| = \left|\frac{m_0 v}{\gamma m_0 v} - 1\right| = \left|\frac{1}{\gamma} - 1\right|$$

Since γ is always greater than one, the fraction is always less than one and we can get rid of the absolute value symbols:

$$\left|\frac{p_c - p}{p}\right| = 1 - \frac{1}{\gamma} < 1 - \frac{3\sqrt{11}}{10} = 0.00501/dots < 1\%$$

Alternatively, as before, since v/c is small we can use the Taylor expansion:

$$1 - \frac{1}{\gamma} = 1 - \frac{1}{1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \frac{15}{48}\frac{v^6}{c^6} + \cdots} \approx 1 - \frac{1}{1 + \frac{1}{2}\frac{v^2}{c^2}} \approx 1 - \left(1 - \frac{1}{2}\frac{v^2}{c^2}\right) = \left(\frac{1}{2}\frac{v^2}{c^2}\right)$$

$$\left|\frac{p_c - p}{p_c}\right| \approx \frac{1}{2}\frac{v^2}{c^2} < \frac{1}{200} < 1\%$$

Alternatively, if we did the % error calculation with the classical term in the denominator:

$$\left|\frac{p_c - p}{p_c}\right| = \left|\frac{p}{p_c} - 1\right| = \left|\frac{\gamma m_0 v}{m_0 v} - 1\right| = |\gamma - 1|$$

Since γ is always greater than one, this is always positive and we can eliminate the absolute value symbols, and put in the Taylor series expansion:

$$\left|\frac{p_c - p}{p_c}\right| = \left|\frac{p}{p_c} - 1\right| = \left|\frac{\gamma m_0 v}{m_0 v} - 1\right| = |\gamma - 1| = \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \frac{15}{48}\frac{v^6}{c^6} + \cdots$$

$$\left|\frac{p_c - p}{p_c}\right| \approx \frac{1}{2}\frac{v^2}{c^2} = \frac{1}{200} < 1\%$$

(b) Show that when $v/c > 99/100$, then [10]

i. $K/m_0c^2 > 6$, and

Solution: The kinetic energy K is related to the rest mass energy so that:

$$\frac{K}{m_0c^2} = \frac{\gamma m_0c^2 - m_0c^2}{m_0c^2} = \frac{m_0c^2(\gamma - 1)}{m_0c^2} = \gamma - 1$$

Given that $v/c > 99/100$ we have:

$$\gamma - 1 = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} - 1 > \frac{1}{\sqrt{1-\frac{9801}{10000}}} - 1 = \frac{1}{\sqrt{0.0199}} - 1 = 7.0888\ldots - 1 = 6.0888\ldots > 6.$$

Thus we have:

$$\frac{K}{m_0c^2} > 6.$$

ii. the relativistic relation $p_0 = E/c$ for the momentum of a zero rest-mass particle may be used for a particle of rest mass m_0 with an error of less than 1%.

Solution: For all particles the relationship between energy and momentum is given by:

$$E^2 = c^2p^2 + m_0^2c^4$$

Solving for p and setting $m_0 = 0$ gives us the momentum for a zero rest-mass particle as stated in the problem:

$$p = \sqrt{\frac{E^2}{c^2} - m_0c^2} \quad \Longrightarrow \quad p_0 = \frac{E}{c}$$

To calculate the % error we take:

$$\left|\frac{p_0 - p}{p}\right| = \left|\frac{p_0}{p} - 1\right| = \left|\frac{\frac{E}{c}}{\sqrt{\frac{E^2}{c^2} - m_0c^2}} - 1\right| = \left|\frac{E}{\sqrt{E^2 - m_0c^4}} - 1\right|$$

For $m_0 > 0$ the total energy is given by

$$E = \gamma m_o c^2$$

$$\left|\frac{p_0 - p}{p}\right| = \left|\frac{\gamma m_0c^2}{\sqrt{\gamma^2 m_0^2c^4 - m_0c^4}} - 1\right| = \left|\frac{\gamma}{\sqrt{\gamma^2 - 1}} - 1\right|$$

The numerator is greater than the denominator in the fraction, so we can eliminate the absolute value symbols since the fraction is greater than one:

$$\left|\frac{p_0 - p}{p}\right| = \frac{\gamma}{\sqrt{\gamma^2 - 1}} - 1$$

$$\gamma^2 - 1 = \frac{1}{1-\frac{v^2}{c^2}} - 1 = \frac{1 - \left(1 - \frac{v^2}{c^2}\right)}{1 - \frac{v^2}{c^2}} = \frac{\frac{v^2}{c^2}}{1-\frac{v^2}{c^2}}$$

$$\frac{1}{\gamma^2 - 1} = \frac{1 - \frac{v^2}{c^2}}{\frac{v^2}{c^2}}$$

$$\frac{1}{\sqrt{\gamma^2 - 1}} = \frac{\sqrt{1 - \frac{v^2}{c^2}}}{\frac{v}{c}}$$

$$\frac{\gamma}{\sqrt{\gamma^2-1}} = \frac{\sqrt{1-\frac{v^2}{c^2}}}{\frac{v}{c}} \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{c}{v}$$

$$\left|\frac{p_0 - p}{p}\right| = \frac{\gamma}{\sqrt{\gamma^2-1}} - 1 = \frac{c}{v} - 1 < \frac{100}{99} - 1 = \frac{1}{99} \approx 1\%$$

Alternatively if we calculated the % error with the zero rest-mass momentum in the denominator we get:

$$\left|\frac{p_0 - p}{p_0}\right| = \left|1 - \frac{p}{p_0}\right| = \left|1 - \frac{\sqrt{\frac{E^2}{c^2} - m_0 c^2}}{\frac{E}{c}}\right| = \left|1 - \frac{\sqrt{E^2 - m_0 c^4}}{E}\right|$$

$$\left|\frac{p_0 - p}{p_0}\right| = \left|1 - \frac{\sqrt{\gamma^2 m_0^2 c^4 - m_0 c^4}}{\gamma m_0 c^2}\right| = \left|1 - \frac{\sqrt{\gamma^2-1}}{\gamma}\right|$$

As calculated above, the term with the γs is equal to v/c, which is less than one, so we can drop the absolute value symbols and proceed:

$$\left|\frac{p_0 - p}{p_0}\right| = 1 - \frac{\sqrt{\gamma^2-1}}{\gamma} < 1 - \frac{99}{100} = \frac{1}{100} = 1\%$$

Alternatively one could start with the relationship for the first part of the question since $E = K + m_0 c^2$:

$$\frac{K}{m_0 c^2} > 6 \quad \Longrightarrow \quad E > 7 m_0 c^2$$

$$\left|\frac{p_0 - p}{p}\right| = \left|\frac{E}{\sqrt{E^2 - m_0 c^4}} - 1\right| < \frac{7 m_0 c^2}{\sqrt{7^2 m_0^2 c^4 - m_0 c^4}} - 1 = \frac{7}{\sqrt{49-1}} - 1$$

$$\left|\frac{p_0 - p}{p}\right| < \frac{7}{\sqrt{48}} - 1 = 0.01036\ldots \approx 1\%$$

Headstart for next week, Week 02, starting Monday 2004/09/20:
– Read Chapter 1 “Relativity” in “Simple Nature” by Crowell
– Read Appendix A “The Special Theory of Relativity” in Eisberg & Resnick
– Read Chapter 1 “Thermal Radiation and Planck’s Postulate” in Eisberg & Resnick