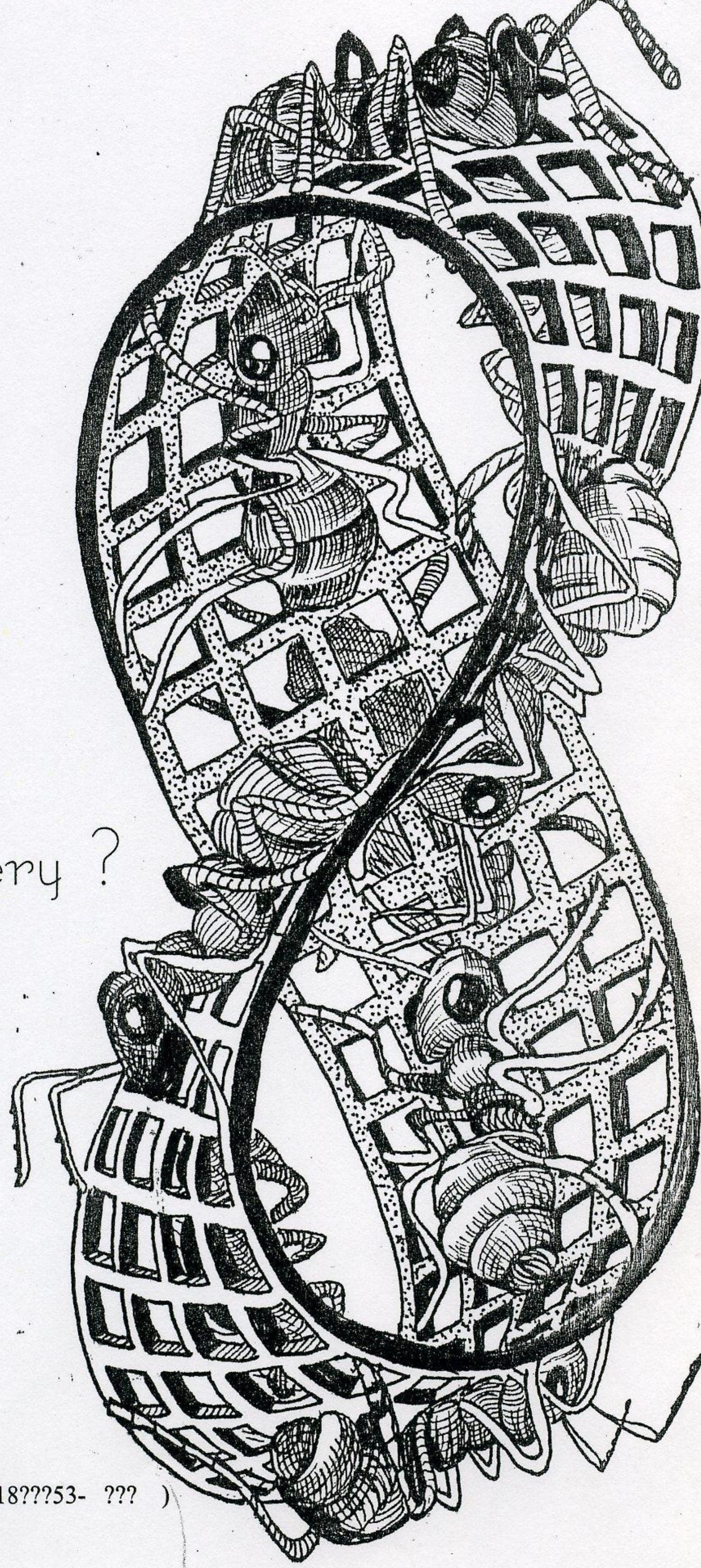
Topology or Tomfoolery ?

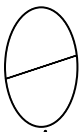
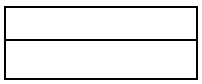


Drawing by E. C. Escher (18???53- ???)

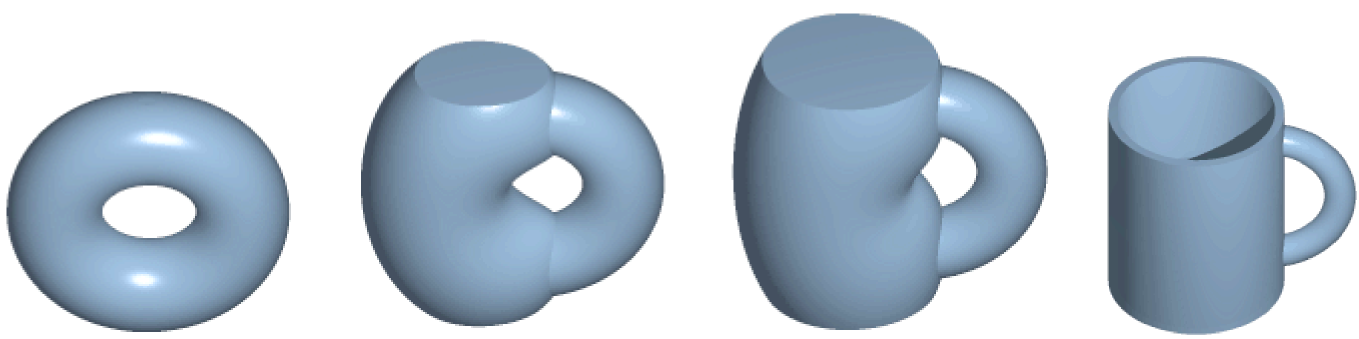
TOPOLOGY or TOMFOOLERY?

Topology (sometimes nicknamed ‘rubber sheet geometry) is the branch of geometry that is concerned with those properties of a thing that are not destroyed through bending, stretching, and twisting, the three specific elements of a topological transformation.

Examples:

 and  are topologically equivalent because you can stretch, bend and twist the first shape into the second one…

..... and so are a doughnut and a coffee mug ! (because you can press a doughnut shape into a coffee mug shape - try it with plasticine)



(adapted from http://en.wikipedia.org/wiki/Topology)

THE IMPOSSIBLE PAPER FOLD

Can you fold a single sheet of paper to create the following item without any cutting off, or attaching of, any pieces?

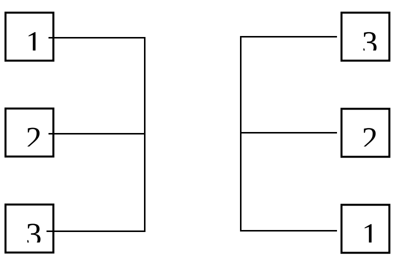
[The solution is in Appendix D, as well as a challenge to go further.]

In its concentration on the unchanging quality properties of things, topology complements the more widely familiar metric geometry, which is concerned with the precise measurements of an object's angles, width, breadth, and so on. That is the geometry that we all study in school and that was original developed by the Egyptians more than 2500 years ago for surveying and architectural purposes.

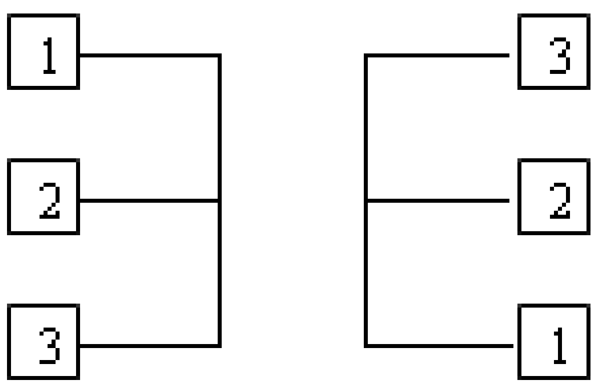
To the metric geometer, for whom the size of things is important, a widening circle represents change. The metric geometer calls our attention to the circle's increasing circumference. But the topologist ignores the superficial changes in the circles appearance and notes that throughout the widening process, the circle remains a simple closed curve with an obvious inside and outside.

This idea of an inside and an outside helps to solve problems of the following type:

"In the diagram below, can you connect 'like numbers' with lines that do not cross each other?"



Practice area



[Example solution: *Join the 1's and the 2's to form a simple closed curve (shade the interior) with one '3' inside and one '3' outside.*] Hence, topology declares the problem to be insoluble.

There is a very old problem that students have suffered with for decades that asks if you can connect up each of the three houses [A, B, and C] to the three utilities [Gas, Water, and Hydro] WITHOUT crossing any utility lines.

A B C

G W H

Try it; use the idea of "inside" and "outside" to show that it actually can't be done on a flat piece of paper.

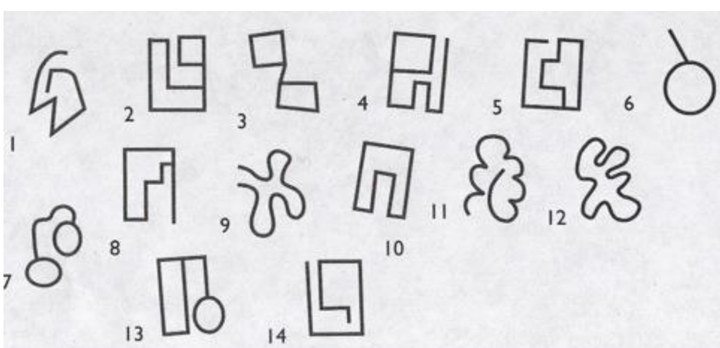
"A child's ... first geometrical discoveries are topological .... if you ask him to copy a square or a triangle, he draws a closed circle."

- Jean Piaget

"A topologist is one who doesn't know the difference between a doughnut and a coffee cup."

- John L. Kelley

But going back to 'topological invariance for a moment, which of the following 'squiggles' are topologically equivalent i.e., which, if you bend, stretch or twist them are actually the same?



And then try to find the letters in the alphabet (shown on the right) that are topologically equivalent (i.e., *homeomorphic*) to the letter E.

MOEBIUS STRIP

The Möbius (or Moebius) band is named after the German astronomer and pioneer topologist Augustus Ferdinand Möbius (1790-1868) who first described this shape.

Take a one strip of paper and give one end of it a half-twist (so that the top side faces down).
Join the ends and gluing them together. It show end up looking like …

Take a pencil and start drawing a line ALONG THE LENGTH OF THE STRIP right down the centre of the strip. Keep going until you come to the end of the line..... what did you notice?
How many 'sides' does the strip have?
How many 'edges' does the strip have?
How many 'sides' does a regular strip loop have?
How many 'edges' does a regular strip loop have?

Now look at the strip that you made and examine the shape "inside" the loop; is it circular?

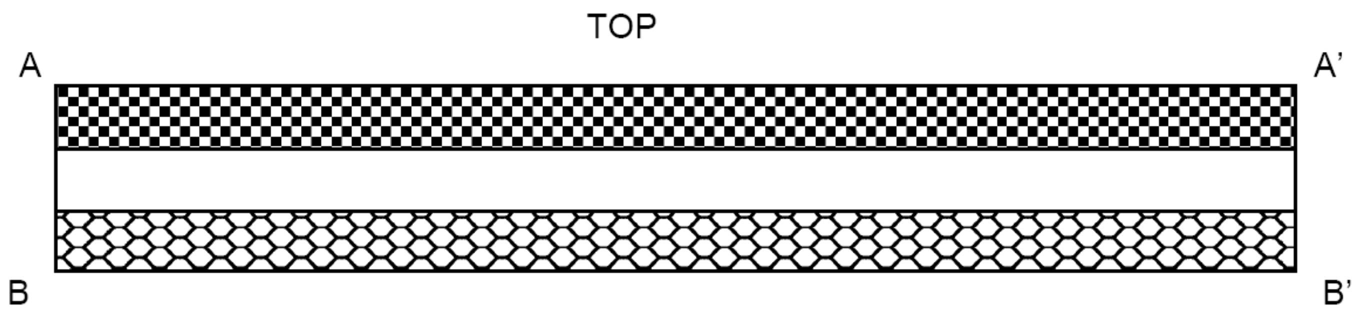
So the question arises, "What shape of paper strip, when formed into a Moebius strip, will provide a circular interior?" Experiment with paper for a while.

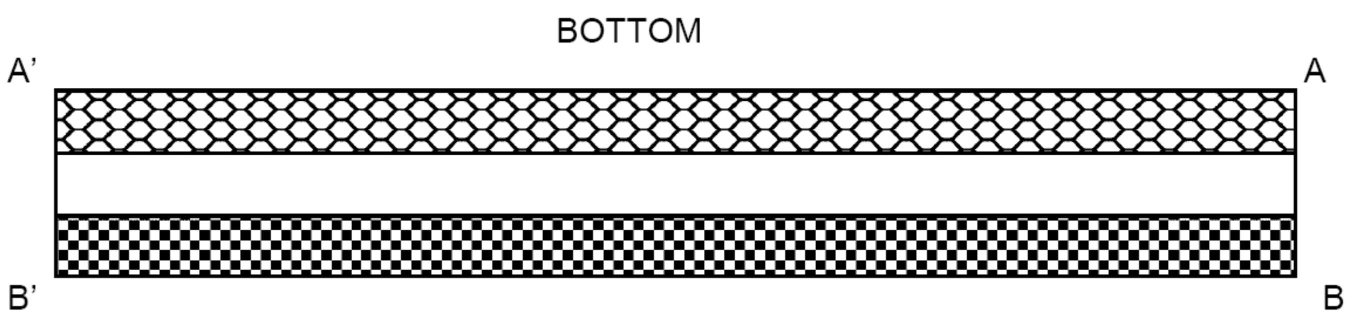
Well, let's stay with this unusual shape for a bit longer.

(i) Take a pair of scissors and cut along the line that was drawn along the centre of your Moebius strip.

> What is the result? How many loops are created? How many sides does each loop have? How many edges? How long are the loops compared to the original loop?
>
> At some other time you can try this with a variety of twists (see Appendix A)

(ii) Take a strip (graph paper 6 or 9 columns wide is best for this). Draw lines lengthwise on both sides a third of the way in from each edge creating lengthwise sections either 2 or 3 columns wide. Colour the top sections using two different colours for the outside thirds (you can leave the middle as is to save time). Now colour the bottom the same way BUT in reverse colour order so that the same colour is NOT on top and bottom. (*see illustration below*).





Take the multi-coloured strip and form a Moebius strip; the colours should match when you join the ends. Notice that the strip has been divided into thirds by the dividing lines.

a) Cut along the dividing lines. What is the result? How many sides does each strip have?

b) Now spread the longer strip along the shorter one until the shorter one is 'sandwiched' by the longer strip. How many colours are visible? Can you reverse the colour of the 'sandwich'?

c) Doesn't it strike you as odd that a 'one-sided' strip can somehow be coated by a 'two-sided' strip?

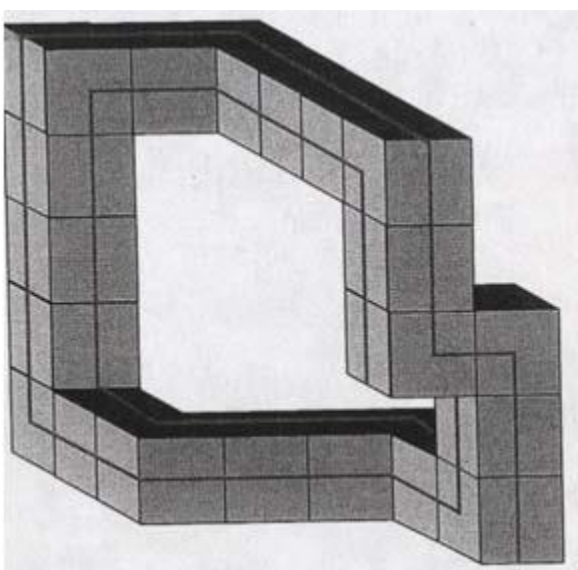
(iii) Take two strips of paper and place them one on top of the other. Create a Moebius band with this double thickness, joining the 'top' layers together and then the 'bottom' layers together. Take a pencil and insert the pencil between the two layers of the band and move it along the band. Does it go all the way around? Would you agree that the two layers are quite separate?

Separate the two layers to get the two separate bands.

How did you make out? Any surprises?

(iv) Open out your double band.
(a) How many sides does it have?
(b) How many half-twists does it have?
(c) Reshape it back into a double-thickness Moebius band.

Now consider this object that seems to have 4 sides but in fact is a Moebius strip only having one side!



It's made up of 22 Link-Cubes:

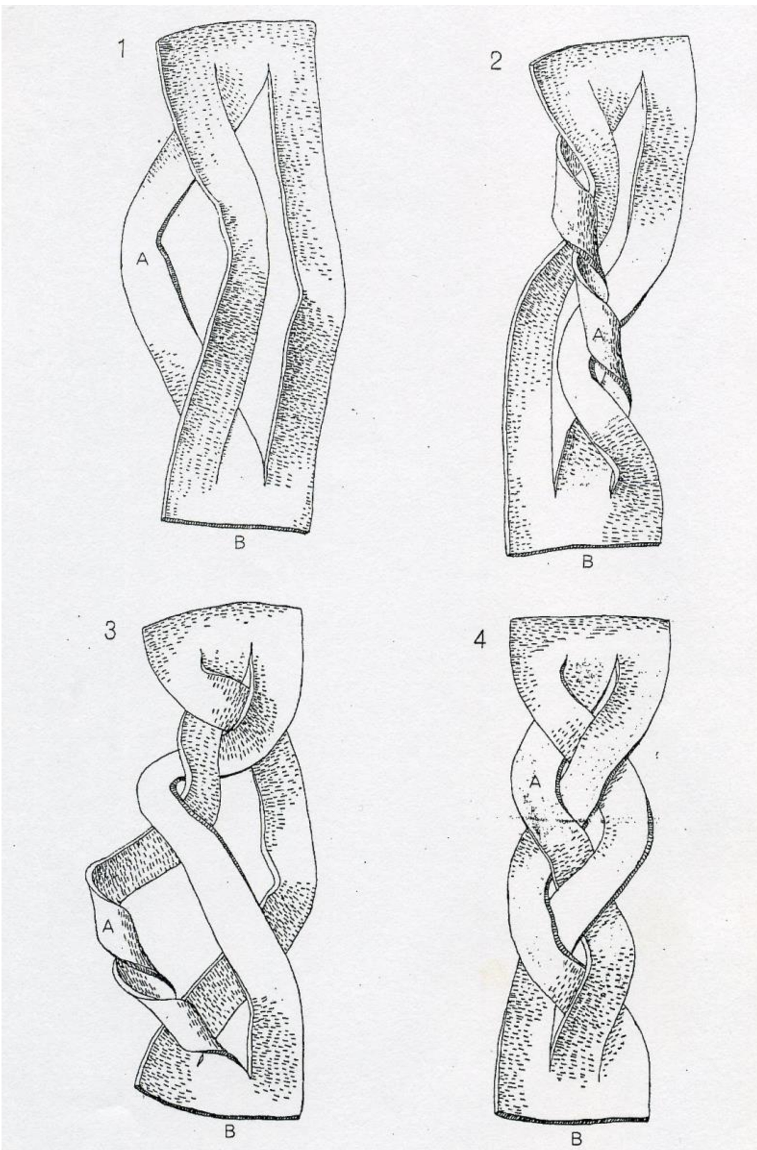
a) Trace a path along its one side (see the line in the illustration) with narrow masking tape.

b) What's the fewest number of cubes you can use to make such a figure?

BRAIDS

Braiding hair is probably as old an activity as mankind itself and finds itself studied in a branch of mathematics called Group Theory. For our purposes, we want to look at one particular braiding topologically. Consider a standard braid that a girl might use for her hair. It's amazing but true that you can take 3 strips joined at both ends and still braid them.

Pull the middle strand back and out to the left; then pull it over the entire bottom end. [*The result is pretty messing looking in the middle.*] Next, take hold of the bottom of the middle strand and pull it out front and to the left; then pull it over the entire bottom end. Flatten out the strands to remove any twists and you're done!



Slade's first trick. Pull A back to front over B, then pull A to left and front to back over B

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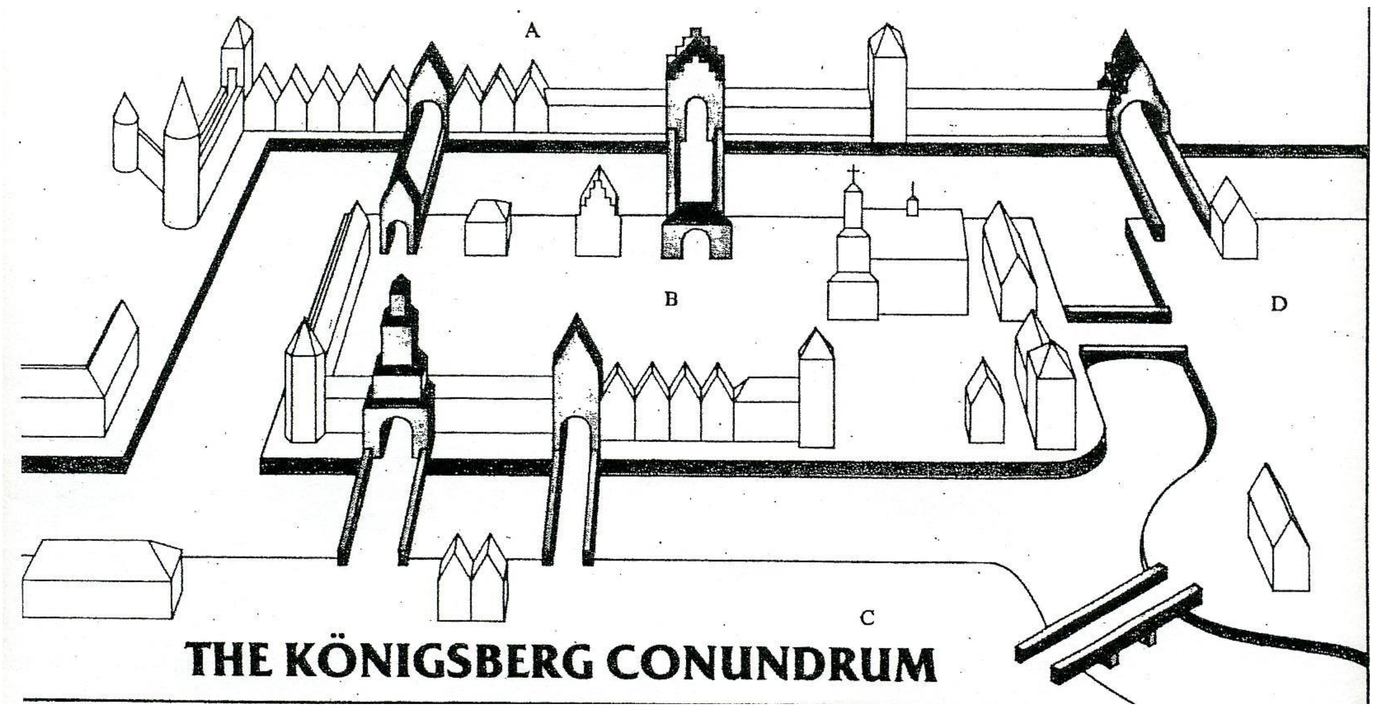
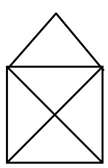
from Scientific American Feb. 1962

NETWORKS

One of the branches of topology, network theory, got a kick start in 1735 when the Swiss mathematician Leonhard Euler was given the now famous problem of the Seven Bridges of Königsberg.
At that time, in the town of Königsberg in Prussia (now Kaliningrad in Russia), there was an island with two branches of the river Pregel flowing around it. There were seven bridges variously connecting the island and the two river banks. Local townsfolk were fond of walking about the island and over the bridges and as Euler described it , “The question is whether a person can plan a walk in such a way that he will cross each of these bridges once but not more than once.”



Well, Euler ended up developing a set of rules that allow anyone to solve this or any other kind of ‘travelling problem’. Into this category fall many of the little problems that we all met as children in school such as ...



Can you draw this figure with one continuous line without lifting your pencil off the paper or retracing any line?

Each corner is called a ‘vertex’, so in this drawing there are five ‘vertices’.
The whole drawing is called a ‘network’.

Euler discovered the following about whether or not a network figure could be so drawn or 'travelled'.

Definitions:
a) a vertex is considered "even" if the number or arcs [or lines] which meet at that vertex is even. Likewise, an "odd" vertex is one ...
b) a network is "travelled" by passing along each of the arcs exactly once.

The following are a summary of Euler's rules:
(i) If a network has more than two odd vertices it cannot be travelled by a single path i.e., you'd have to lift your pencil or have to go over a previous line.
(ii) If a network has no odd vertices it can be travelled starting at any point and returning to that point.
(iii) If a network has two odd vertices it can be travelled only by starting at one odd vertex and returning to the other odd vertex.

So, can the 'house' network be 'travelled' after all? Well …
a) how many vertices are there?
b) how many of them are 'odd'?
c) what do Euler's rules say about this situation? Try it!
d) try it again but this time NOT following Euler's rules.

So how does any of this help answer the question about the Seven Bridges of Königsberg? Could they be 'travelled'?

To answer the problem we redraw the river, bridges and land by representing the land parts as dots, and the bridges as lines (or arcs) joining the dots. There are four land parts labeled A, B, C, and D in the picture above. For every bridge, draw an arc joining the two land parts that the bridge joins.

- There are 2 bridges from A to B
- There are 2 bridges from C to B
- There is 1 bridge from D to A
- There is 1 bridge from D to B
- There is 1 bridge from D to C

A●

So a representation of the situation could be …

B● ●D

So, can the Bridges be 'travelled'?

C●

Likewise, can the mouse (see picture below) eat all the vegetables by passing through all of the doors ONCE ONLY and exiting at the bottom (arrow) door?

Try your new technique by representing the room by dots and the doors by lines and then applying Euler's rules.

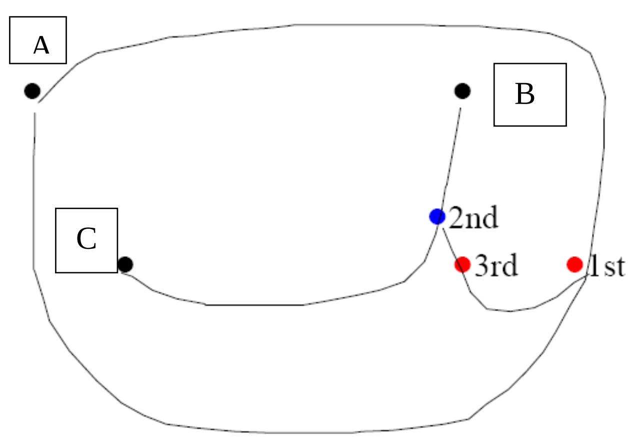
'SPROUTS'

In the 1960's a topologically based game, called 'Sprouts', became a craze at Cambridge University. It was created jointly by a mathematics professor, John Horton Conway, and a graduate student, Michael Stewart Patterson.

The game begins with a number of dots on a sheet of paper; their arrangement is irrelevant. (Even with only 3 dots (**recommended**) the game is much more difficult to analyze than X's and O's.) Players alternate taking turns.

The rules are:
(i) A move consists of joining a dot to a dot (**it can be the same dot**) AND placing a new dot anywhere on the newly drawn line (its position is irrelevant).
(ii) The line drawn can have any shape but it cannot cross itself or any other line.
(iii) No dot may have more than 3 arms going out from it; *hence the 1st and 2nd dots are now eliminated*.
(iv) Normally, the last person to make a legal move is the winner. However, for a change, you can play the "misère" version in which the last legal move is the losing one.

Sample game beginning:
(Starting with 3 dots, one possibility for the first three moves is shown.)
(i) A to A, (ii) B to C (iii) 2^{nd} to 1^{st}

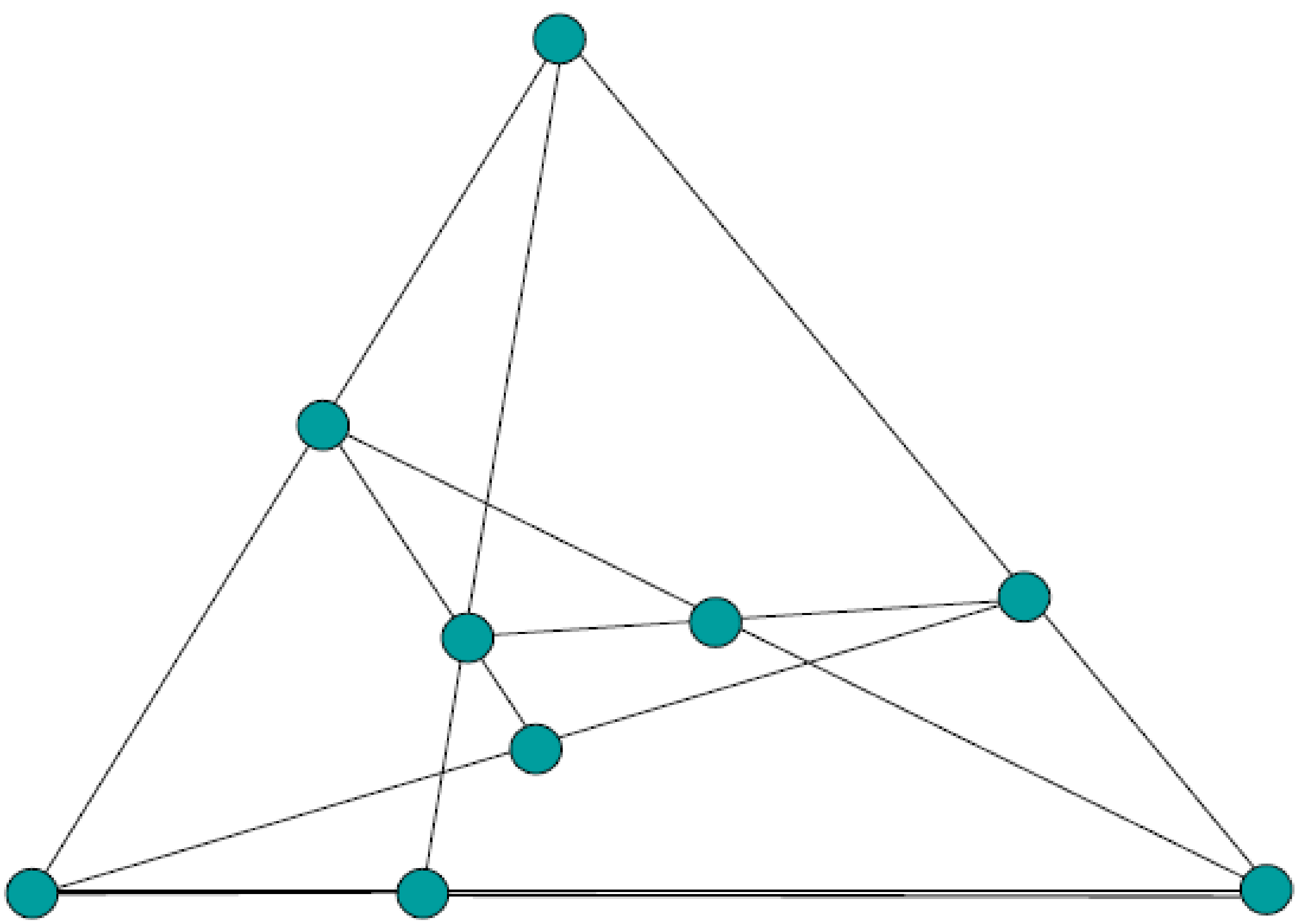


'TRI-HEX'

Tic-Tac-Toe or 'X's and O's' is a much mo challenging game without a trivial solution when played on the pattern below and was named by Thomas H. O'Heirne of Glasgow as Tri-Hex.
Each player has four distinct counters (e.g., pennies and dimes) only; no fifth move is permitted player #1.
As with X's and O's, players alternate placing one counter on the pattern in an attempt to make 3-in-a-row.

Question: if both players make their best moves possible, does player #1 win, player #2 win, or is it always a draw?

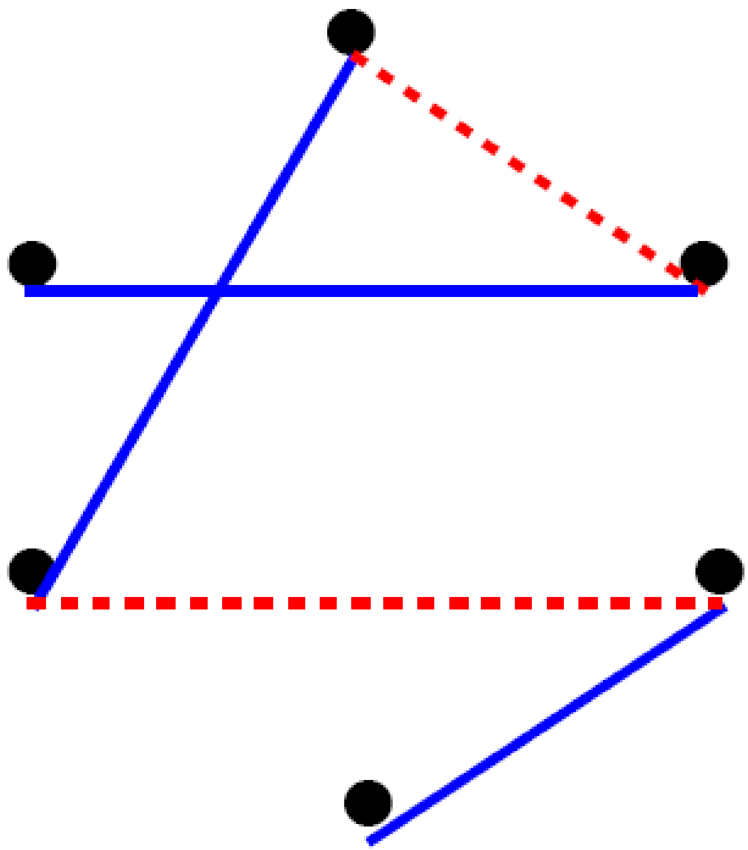
Note: a larger version of this game board is in Appendix E.



THE GAME OF 'SIM'

Six points are marked on a sheet of paper (usually as the vertices of a mostly regular hexagon; not required). Two player take turns, using two different colours, joining any two of the original six vertices. Two vertices may only be joined once.
The first player forced to form a triangle of his own colour (only triangles whose vertices are among the original six count) is the loser.

For example, here is the game position after BLUE (solid) has made 3 moves and RED (dotted) has made 2 moves; it is now RED's turn to play.

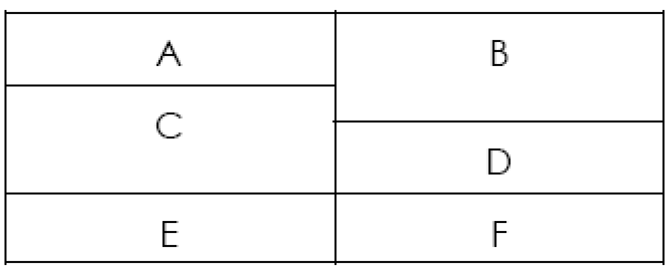


FOUR-COLOUR ~~PROBLEM~~ THEOREM

One of the most famous problems in mathematics which is related to networks and regions was the 4-colour problem. In the ‘90s, a solution was discovered using a computer. However, because the computer program was so large and complicated very few people were able to get a sufficient handle on it to try to decide if the program actually provided a legitimate proof.

The problem was to determine if it was possible to colour any map drawn on a plane surface (i.e. in 2 dimensions) with only four colours so that no two neighbouring countries had the same colour. Countries are considered to be ‘neighbouring’ if they share any portion of a border other than a single point (i.e. a vertex).

For example, countries C and B are ‘neighbours’ but F and C are not.

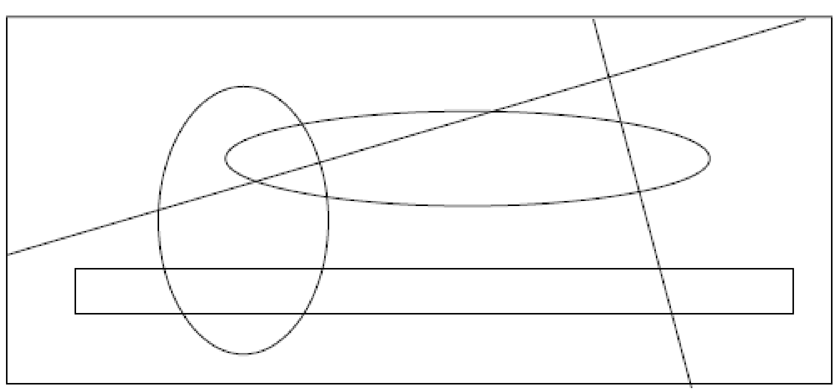


‘Fun’ activity: try drawing a map that forces a fifth colour to be used.

A few interesting problems arise from this concern with adjacent areas:

BLACK AND WHITE?

How many colours are needed to colour a map drawn with lines that either cut across or form closed curves?

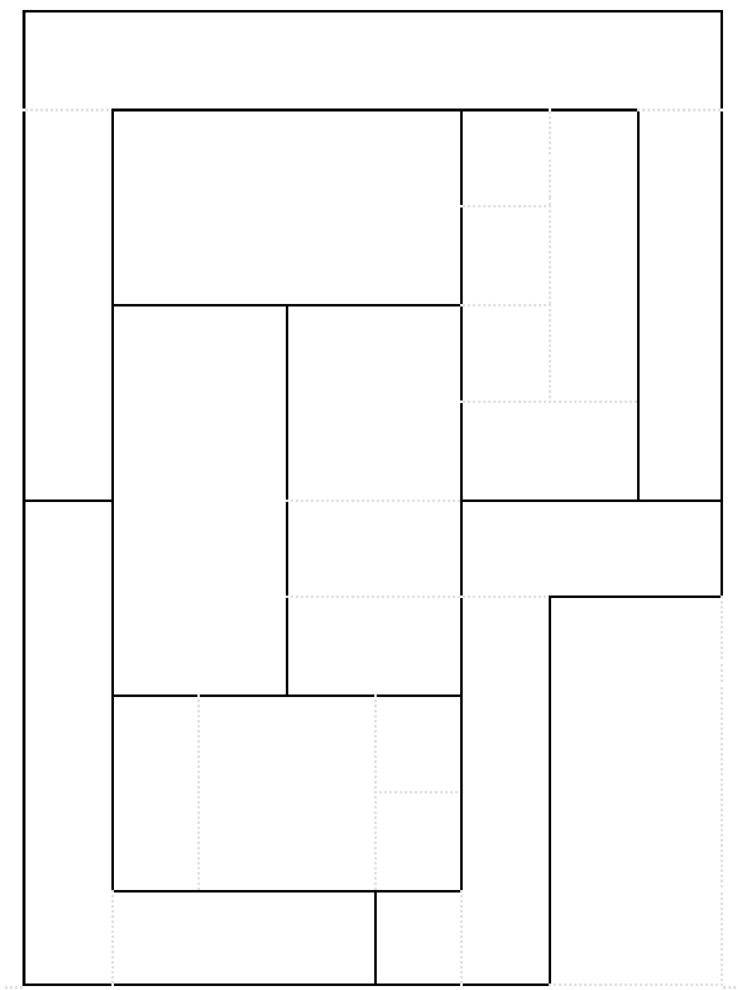


THE ARTIST'S DILEMMA

Stephen Barr, in his book "Experiments in Topology" writes about the painter who wished to complete on a huge canvass the non-objective work of art shown in the outline below. He decided to limit himself to four colours, and to fill each region with one solid colour in such a way that no two bordering regions would have the same colour.

Each region had an area of 8 square feet, except for the top region which was twice the size of the others. When he checked his paint supplies, he found that he only had the following: enough red to cover 24 square feet, enough yellow to cover the same area, enough green to cover 16 square feet and enough blue to paint 8 square feet.

How did he manage to complete his canvass with only the paint that he had? (*hint: you really have to be a creative artist!*)



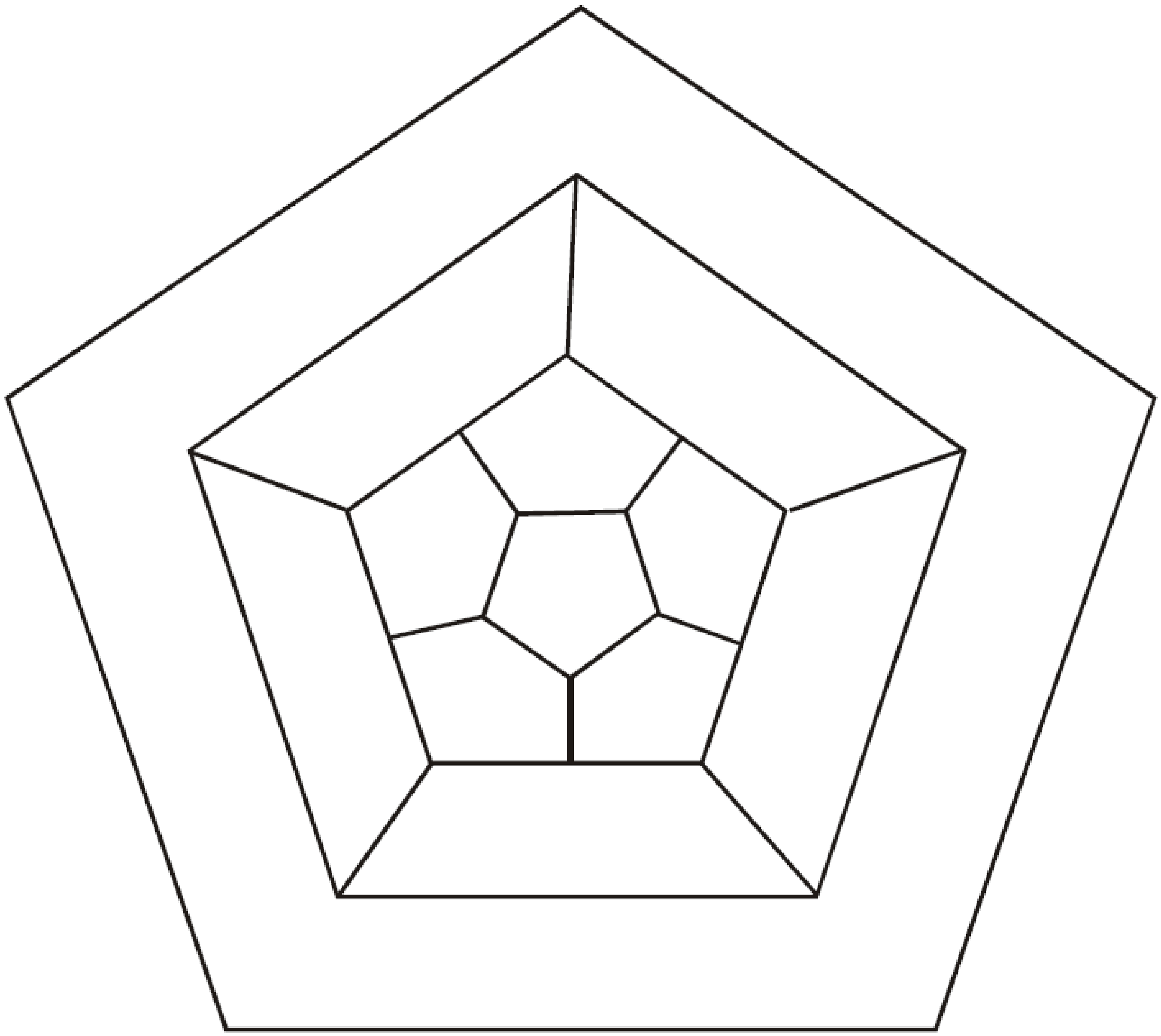
From "Experiments in Topology" by Stephen Barr, 1964

BRAMS'S GAME

Drawing from a **supply of five colours**, two players alternate 'colouring' a section so that no two adjacent sections have the same colour. **BUT**, the two players have different objectives!

Player 1 aims to have the 'map' coloured with five or fewer colours.
Player 2 aims to force the need for a sixth colour.

Whoever achieves his/her goal wins the game.
One of the players *should* be able to always win ... but to find who, and what that strategy is, is the second part of this problem; first, simply try to win.



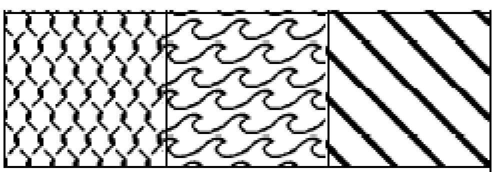
MAP FOLDING

"Ever since the 20th century Polish mathematician Stanislaw Ulam first posed the question of how many different ways a map can be folded, the problem has frustrated researchers in the field of combinatorial theory.
The difficulty arises from the fact that even the simplest map – or any rectangular piece of paper - has many possible ways of being folded.
There is an old saying that states, "The easiest way to fold a map is ... differently."

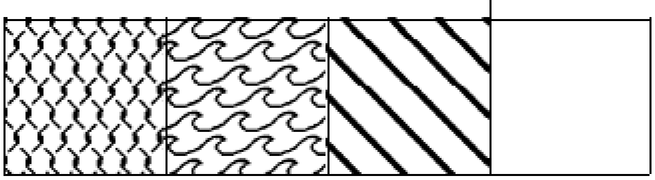
3 PROBLEMS + 1 'BAD' Problem

How many different ways can you find to fold ...

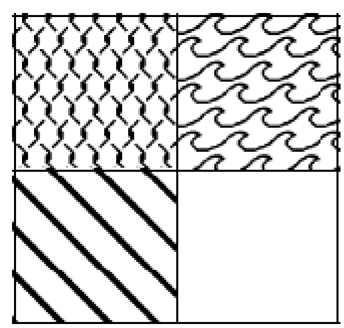
a) a three-square paper strip



b) a four-square paper strip



c) a four-square paper square



The squares are the same colour on both sides,
so it doesn't matter which side is up in the final arrangement.

[See Appendix F for a blank master.]

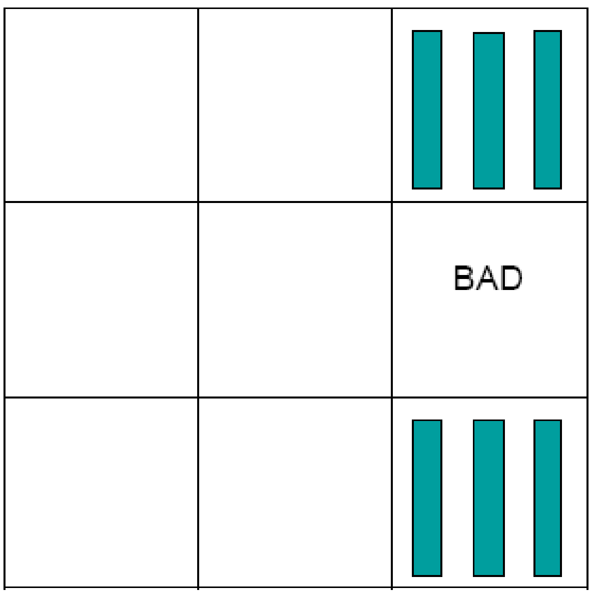
Note: if the squares were 'lettered' A,B,C,D then having a folded order BACD would be considered the same as DCAB, which you would get by simply turning the original arrangement upside-down.

A 'BAD' PROBLEM:

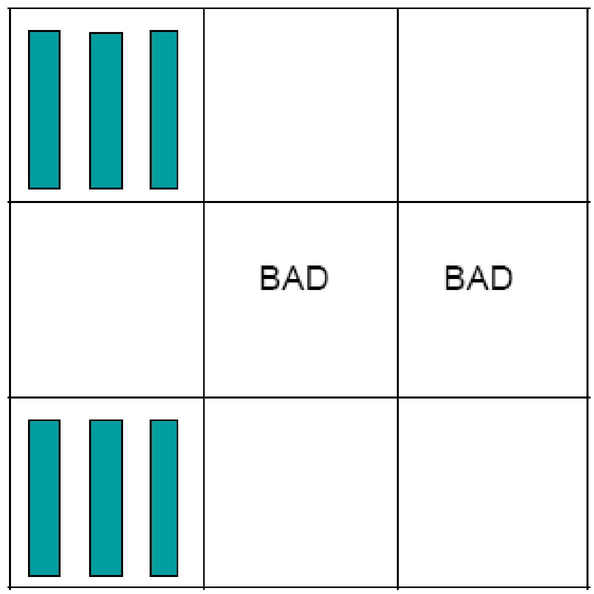
A 3 x 3 paper grid has pictures/names (or just the word **BAD**) in three cells. [The original had pictures of Hitler, Stalin and Mussolini.]

The paper grid has also has prison bars in two of the squares - formed by slicing out little strips of paper (no more than 3 strips - to keep it practical) or by using a hole punch to make the spaces between bars (easier than slicing the strips.)

Fold the paper in such a way that TWO of the **BAD** people end up looking out from behind bars.



FRONT



FLIP SIDE

[The solution is illustrated and explained in Appendix B]

KNOTS

“Everyone who can tie his shoes understands a little about knots. But mathematicians have turned knots into a field of deep topological study. [But] don’t expect to untie a mathematical knot: both ends of a mathematicians knot are joined to form an endless loop.

The topology of knots is not merely of interest to recreational and professional mathematicians, it has enormous importance in several other branches of science, particularly molecular biology” as in the study of the DNA molecule and complexly folded proteins.

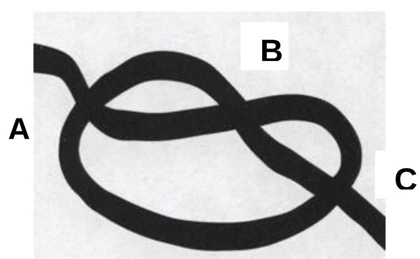
CREATING AN IMPOSSIBLE OVERHAND KNOT

The challenge is to take hold of the ends of a length of rope, one in each hand, and without letting go of the rope or cutting it, tie a single overhand knot in the rope.

[The solution is shown in Appendix C.]

SHADOW KNOT

You see a length of rope lying on the floor in front of you in a very dimly lit room. The rope is twisted somehow but it is too dark to tell whether the rope passes over, under or through the loop. Depending on how the rope is lying, pulling on the ends will either straighten the rope or tighten a knot in the rope.



What is the probability that the rope is actually knotted rather than just coiled?
[Hint: it depends on how many ways the rope can lie at points A, B, and C. The answer is in Appendix C.]

KNOTTY PROBLEMS

#1 Button hole problem

#2 Paperclips on a folded piece of paper
- The clips go over two of the sections and are placed closest to the free ends.

#3 Handcuffs

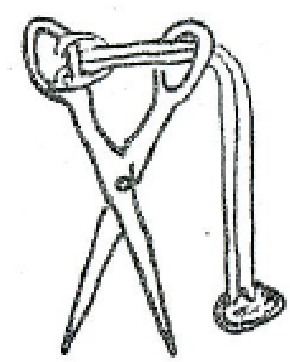
#4 Attach scissors to a handcuff cord

#5 Paper with strings and tags puzzle

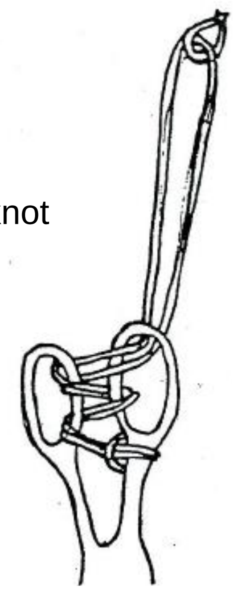
#6 Vest reversal

#7 Keyhole puzzle (block with 2 beads)

#8 Scissors and button knot



#9 Gordian knot



APPENDIX A

The Moebius Strip - Investigation

Number of half-twists in original loop	When original loop cut down centre … number of loops created	Length of first loop	Number of half-twists in first loop	Length of second loop	Number of half-twists in second loop
0	2	same as original	0	same as original	0
1 (Moebius)					
2					
3					
4					

Are there any patterns?

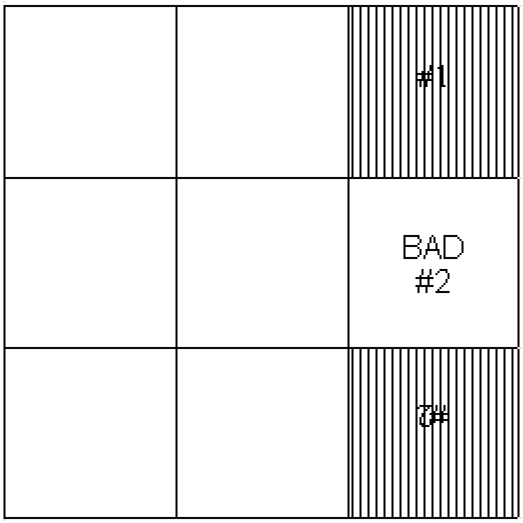
Advanced investigations:
After cutting a strip with twist along its length, cut the resulting loop(s) along their length again.

The Moebius Strip – Making the opening a circle

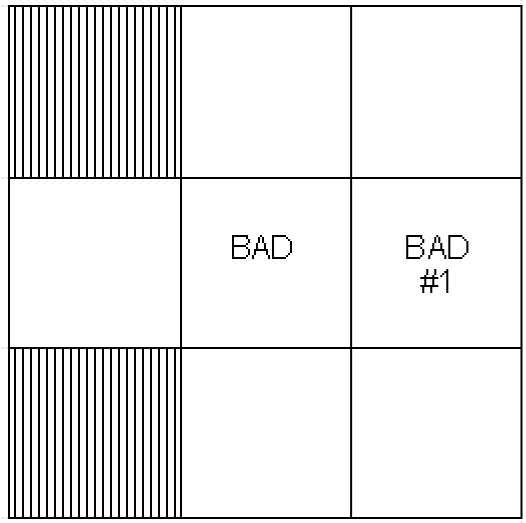
To end up with a circular opening in a Moebius strip, start with a strip in the shape of an S.

APPENDIX B

‘BAD’ Problem SOLUTION

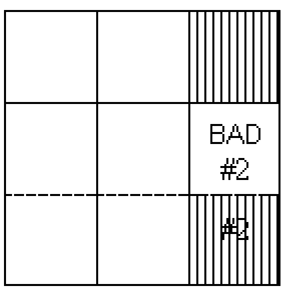


FRONT



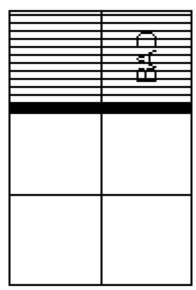
FLIP SIDE

STEPS 1a, 1b



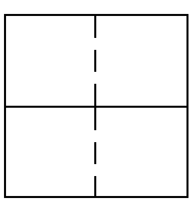
- ☞ **Valley-fold** bottom flap up over the middle row; i.e. BARS#2 onto BAD#2
- ☞ **Rotate** 90 degrees counterclockwise

STEP 2



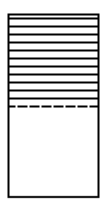
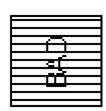
- ☞ **Mountain-fold** the top flap down behind the middle row.

STEP 3



- ☞ **Valley-fold** left flap onto right flap

STEP 4

 to get ... 

DONE!

- ☞ **Valley-fold** the bottom flap up BUT ... **tuck** it under BARS#1

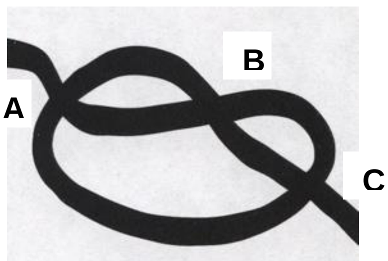
APPENDIX C

CREATING AN IMPOSSIBLE OVERHAND KNOT

The challenge is to take hold of the ends of a length of rope, one in each hand, and without letting go of the rope or cutting it, tie a single overhand knot in the rope.

[The secret is to first cross your arms (thereby making an overhand knot with them) before taking hold of the ends of the rope. By then pulling your arms apart you transfer the knot onto the rope which is simply a continuation of your arms.]

SHADOW KNOT



There are eight possible combinations …

at A – over/under; at B – over/under; at C – over/under.

Of the eight combinations only under/over/under or over/under/over create a knot i.e., the probability is 2 in 8 or, ¼.

KNOTTY PROBLEMS

#1 Button hole problem

#2 Paperclips on a folded piece of paper
- The clips go over two of the sections and are placed closest to the free ends.

#3 Handcuffs

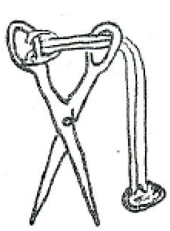
#4 Attach scissors to a handcuff cord

#5 Paper with strings and tags puzzle

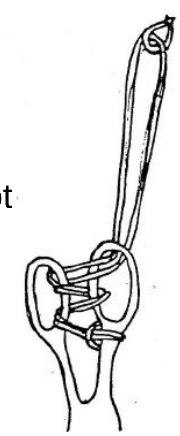
#6 Vest reversal

#7 Keyhole puzzle (block with 2 beads)

#8 Minor Gordian knot



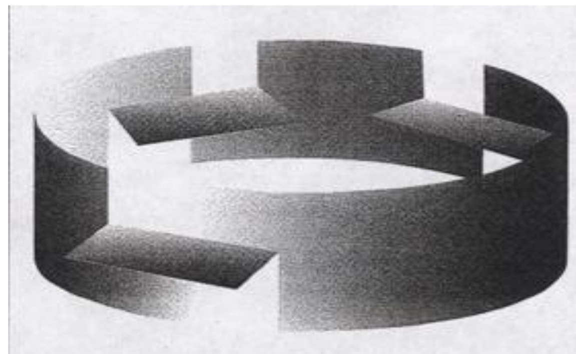
#9 Gordian knot



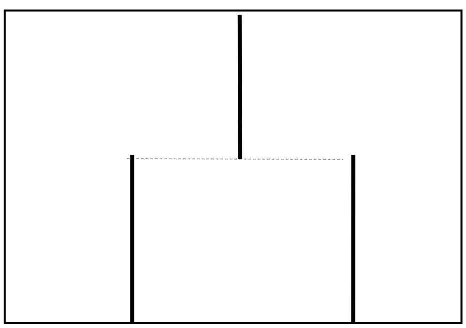
- Move the knot loop carefully along the double chord out and over the button

APPENDIX D

THE IMPOSSIBLE PAPER FOLD



Make the following cuts …just the SOLID lines.



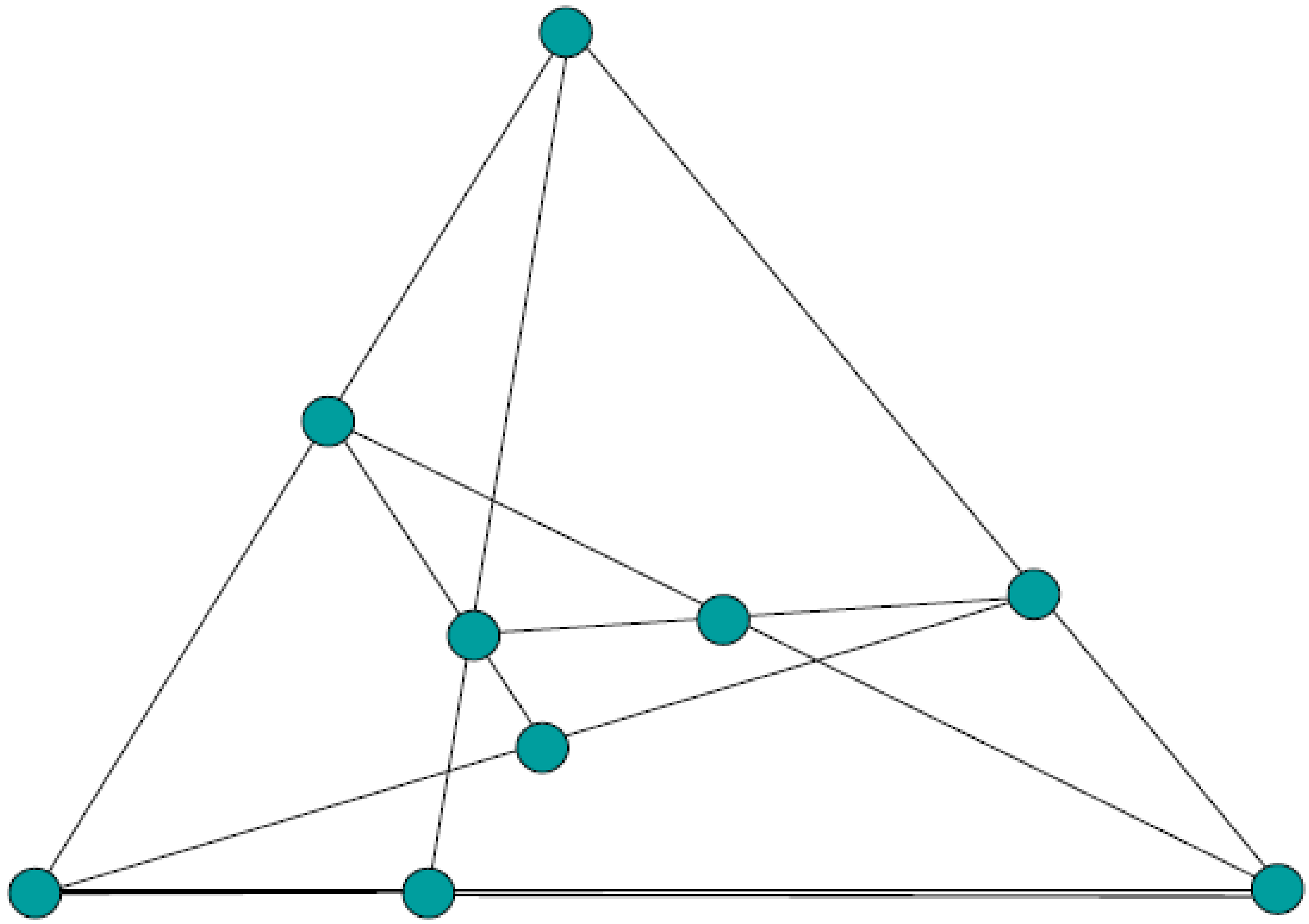
The master for the Impossible Paper Fold is on the last page. CUT on the 3 bold lines and **only FOLD on the dotted line**.

(i) the top vertical cut is in the middle and goes down halfway exactly
(ii) the two bottom vertical cuts are ¼ and ¾ across the paper and go up exactly halfway
(iii) Fold the flap along the dotted line so that it sticks straight up in the air
(iv) Fold section A along the dotted line also, but all the way around so that the top edge is now at the bottom edge and the bottom edge is at the top.

It is a more effective paradox if you now glue the structure onto a piece of cardboard so that others can't manipulate the paper.

APPENDIX E

TRI-HEX



APPENDIX F

MAP FOLDING master

Number/Letter/Colour the squares with different numbers/letters/colours.
The squares are to be the <u>same number/letter/colour on both sides</u>, so it doesn't matter which side is up on the final arrangement.

For example, you could letter the strip … (remember, on BOTH sides)