Physics 380H - Wave Theory — Homework #08 - Solutions

Fall 2004 — Due 12:01 PM, Monday 2004/11/15

[50 points total]

"Journal" questions:
– Of the material that has been covered in the course up to the mid term test, what has been the most difficult for you to understand? What material has been the most interesting? What material has been the most surprising? Is there any material that you thought you understood before this course that you now have a drastically different understanding of? What was is and what has changed?
– Any comments about this week's activities? Course content? Assignment? Lab?

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1. Please complete the anonymous mid-course survey online on WebCT. Early feedback will hopefully allow us to have the best possible course this semester rather than just having next year's students benefit. In addition to the bonus assignment marks, survey participation may count towards overall class participation scores. [5.01-bonus]

   Solution: Do the survey - get the bonus marks.

2. (From Towne P12-1, pg 289) Let θ be the angle in Figure 12-2 which corresponds to the *first* minimum, under the conditions of the Fraunhofer approximation, in the radiation pattern from a coherent line source.

   (a) For this case what is the phase difference at P between the contributions from O, the centre of the course, and C, the lower end? Compare the contributions from other pairs of corresponding points, i.e., other pairs in which the first point lies on the upper half of the source a certain distance above O and the second lies on the lower half the same distance above C. [5]

   Solution: Since this is the *first* minimum, $\beta = \pi \rightarrow \sin\theta_1 = \lambda/a$. Thus the path difference between the contribution from the centre of the course and the lower end of the course would be exactly one half of a wavelength, and therefore the phase difference would be π radians or 180°. Similarly for every pair of points on the line source separated by a distance of $a/2$ where a is the distance $\overline{DC}$. From trigonometry we have that the path difference $\delta = r_1 - r_2 = (a/2)\sin\theta_1 = \lambda/2$, so the angle θ_1 is such that $(a/2)\sin\theta_1 = \lambda/2$ or $a\sin\theta_1 = \lambda$.

   (b) From this information can you predict what the resultant of all the contributions should be? [5]

   Solution: Since each pair of points are destructively interfering, there will be <u>no</u> net resultant at this angle, $I(\theta_1) = 0$. This is a result of the arguments about various points on the source in addition to the direct calculation of the appropriate integral.

   (c) Devise a similar interpretation for the *second* minimum in the Fraunhofer radiation pattern. [5]

   Solution: The second minimum in the Fraunhofer radiation pattern occurs when $\beta = 2\pi \rightarrow \sin\theta_1 = 2\lambda/a$. This corresponds is to the angle θ_2 at which each pair of points along the line source separated by a distance of $a/4$ are completely out of phase. In other words the first quarter of the line destructively interferes with the third quarter of the line, and the second quarter of the line destructively interferes with the fourth quarter of the line. The angle θ_2 is such that $a\sin\theta_2 = 2\lambda$.

   More generally, the n^{th} minimum in the Fraunhofer radiation pattern corresponds to the angle θ_n at which each pair of points along the line source separated by a distance of $a/(2n)$ are completely out of phase. The angle θ_n is such that $a\sin\theta_n = n\lambda$.

3. (From Towne P12-4, pg 290) Sketch a polar plot of intensity vs. angular position which corresponds to the Fraunhofer diffraction of a normally indicent acoustic plane wave by an extremely narrow slit which is three wavelengths in length. [10]

Solution: We have the expression for the Fraunhofer radiation intensity pattern due to a line source of

$$I(\theta) = I_0 \left(\frac{\sin\beta}{\beta} \right)^2 \qquad \beta = \frac{ka}{2}\sin\theta.$$

For $a = 3\lambda$ we have

$$\beta = \frac{ka}{2}\sin\theta = \frac{2\pi 3\lambda}{2\lambda}\sin\theta = 3\pi\sin\theta.$$

So we would like to plot

$$I_1(\theta) = I_0 \left(\frac{\sin(3\pi\sin\theta)}{3\pi\sin\theta} \right)^2.$$

Using Maple and the following commands we get figure 1 and figure 2.

```
> restart; with(plots);
> I_0 := 1; beta := theta->3*Pi * sin(theta);
> I_1 := theta->I_0*(sin(beta(theta))/beta(theta))^2;
> polarplot(I_1(theta), title="I_1(theta)");
> polarplot(I_1(theta), title="I_1(theta)", view=[-1..1,-1..1]);
> polarplot(I_1(theta), title="I_1(theta-tight)", view=[-0.05..0.05,-0.05..0.05],
  numpoints=5000);
```

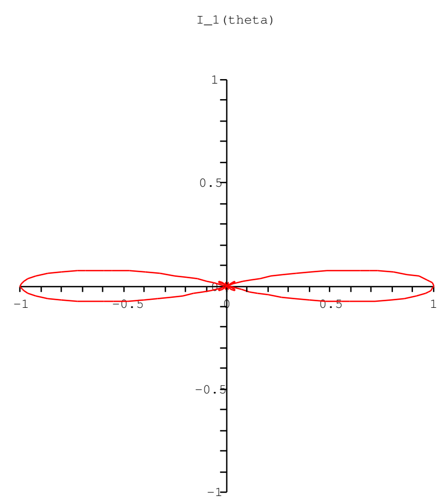

Figure 1: $I_1(\theta)$

To get a feel for what the intensity is like for the small lobes we can view this at a different scale in figure 2 by zooming in on those smaller lobes.

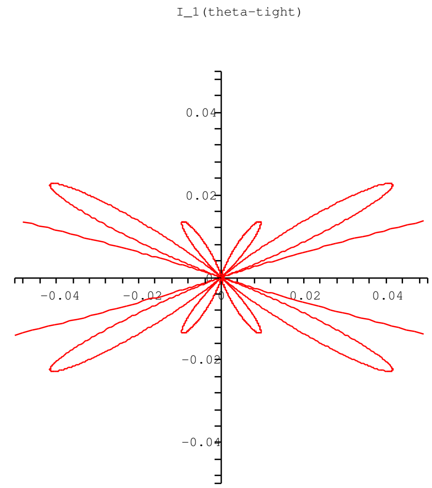

Figure 2: $I_1(\theta)$ - detail

Note how this is similar to the plot that Towne makes in Figure 12-5.

4. (From Towne P12-6, pg 290) Sketch a polar plot of intensity vs. angular position which corresponds to the Fraunhofer diffraction of an acoustic plane wave by an extremely narrow slit which is three wavelengths in length where the angle of incidence of the plane wave is 45°. (Assume that the normal to the plane wave lines in a plane perpendicular to the long dimension of the slit.) [10]

Solution: One might think that this is an identical problem to that above, except for the shifting of the pattern by $\phi = \pi/4\,\text{rad} = 45°$ and a reduction in the effective width from a to $a\cos\phi$, however this would only be true if $ka \gg 1$, which in this case it is not since $ka = 6\pi \approx 19$. So we have to use Towne's more general expression for β, namely

$$I(\theta) = I_0\left(\frac{\sin\beta}{\beta}\right)^2 \qquad \beta = \frac{ka}{2}[\sin\theta - \sin\phi].$$

For $a = 3\lambda$ and $\phi = \pi/4\,\text{rad} = 45°$ we have

$$\beta = \frac{ka}{2}[\sin\theta - \sin\phi] = \frac{2\pi 3\lambda}{2\lambda}[\sin\theta - \sin(\pi/4)] = 3\pi[\sin\theta - \frac{1}{\sqrt{2}}].$$

So we would like to plot

$$I_1(\theta) = I_0\left(\frac{\sin(3\pi[\sin\theta - \sin\phi])}{3\pi[\sin\theta - \sin\phi]}\right)^2.$$

Again using Maple and the following commands we get figure 3 and figure 4.

```
> restart; with(plots);
> I_0 := 1; phi := Pi/4;
> beta_2 := theta->3*Pi * (sin(theta)-sin(phi));
> I_2 := theta->I_0*(sin(beta_2(theta))/beta_2(theta))^2;
> polarplot(I_2(theta), title="I_2(theta)", view=[-1..1,-1..1]);
> polarplot(I_2(theta), title="I_2(theta)-tight", view=[-0.05..0.05,-0.05..0.05],
  numpoints=5000);
```

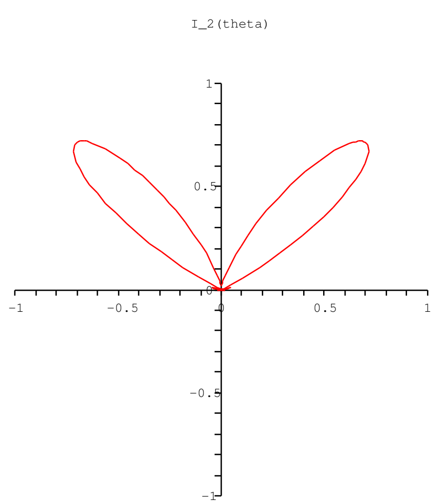

Figure 3: $I_2(\theta)$

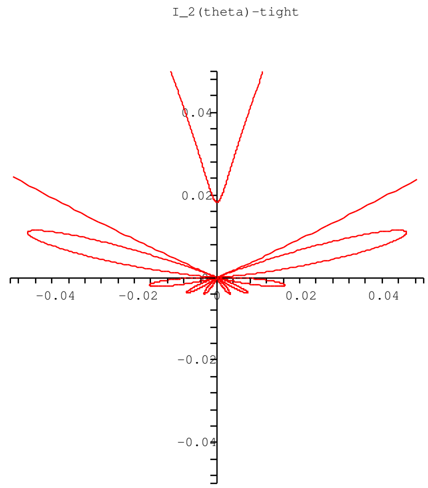

Figure 4: $I_2(\theta)$ - detail

Note that in figure 3, we essentially have a reflection in addition to the transmission of the wave coming in at the 45°angle. This is in contrast to what we would have had by taking β to have had just the $ka \gg 1$ approximation. As Towne states, the $\beta = \frac{ka}{2} \cos\phi \sin(\theta - \phi)$ approximation only works for angles close to $\theta \approx \phi$, and thus does not give this reflection information.

5. (From Towne P12-11, pg 291) Assume that the limiting aperture of an optical system is rectangular rather than circular. For example, suppose that objects are to be viewed by an eye placed directly behind a laboratory slit. Consider two point objects which are aligned parallel to the narrow dimension a of the slit. Assume that the images of two objects are unresolved if there is any overlapping of the central lobes in their Fraunhofer patterns.

   (a) Show that the angular limit of resolution is given by $(\Delta\theta)_{\text{min}} = 2\lambda/a\cos\theta$, where θ is the angle which the objects make with the axis normal to the slit at its centre. [10]

   Solution: For sources that are close to perpendicular to the plane of the aperture, the first minimum of the Fraunhofer radiation intensity pattern is at $\beta = \pi$ for $\beta = \frac{ka'}{2}\sin\phi' \approx \frac{ka'}{2}\phi'$ where ϕ' is measured from the line between the source and the aperture and $a' = a\cos\theta$ is the perpendicular extent of the aperture. For the first minimum therefore we have $\phi' = (2\pi)/(ka') = \lambda/(a\cos\theta)$.

   We are interested in the angle $\Delta\theta$, the angle between two sources such that their first minima are coincident. If the first source is at angle θ_1 and the second is at a slightly greater angle θ_2 and $\Delta\theta = \theta_2 - \theta_1$ we know that the first minima of the first source is at angle $\theta_1 + \phi'$ and the minima of the second source is at angle $\theta_2 - \phi'$. If we want these minima to be coincident we need that $\theta_2 - \phi' = \theta_1 + \phi'$ or $\theta_2 - \theta_1 = \phi' + \phi' = 2\phi'$.

$$\begin{aligned}\Delta\theta &= \theta_2 - \theta_1 \\ &= \phi' + \phi' = 2\phi' = \frac{2\pi}{ka'} \\ \Delta\theta &= \frac{2\lambda}{a\cos\theta} \qquad (5.1)\end{aligned}$$

   (b) Consider a horizontal string of light bulbs spaced 1 m apart along the railing of a bridge. These are viewed from a distance of 1 km through a vertical slit. Apply the criterion above to find the slit width at which resolution would be lost. (Assume that $\lambda = 5500\,\text{Å}$.) [5]

   Solution: Assuming that the railing of the bridge is perpendicular to the where the bulbs are being viewed, $\theta \approx 0$ and $\cos\theta \approx 1$. The angle between the bulbs is given by $\tan(\Delta\theta) \approx \sin(\Delta\theta) \approx \Delta\theta \approx (1\,\text{m})/(1\,\text{km}) = 10^{-3}$ radians. Applying the results of (5.1) we have

$$\begin{aligned}\Delta\theta &= \frac{2\lambda}{a\cos\theta} \\ a_{\text{min}} &= \frac{2\lambda}{\Delta\theta\cos\theta} \\ &\approx \frac{2(5500\,\text{Å})}{(10^{-3})(1)} = 1.1\times10^{-3}\,\text{m}.\end{aligned}$$

   Thus the slit must be at least 1.1 mm wide in order to resolve the individual light bulbs.

Headstart for next week, Week 09, starting Monday 2004/11/15:
– Read Chapter 12 "Continuous Distribution s of Coherent Sources; the Fraunhofer Approximation" in Towne, omit 12-15
– – Section 12-10 "Oblique incidence"
– – Section 12-11 "Reflection of a plane wave from a rectangular surface"
– – Section 12-12 "Fraunhofer diffraction by a circular aperture"
– – Section 12-13 "Acoustic radiation from a circular piston"
– – Section 12-14 "Limit of resolution of image forming instruments"
– Read Chapter 13 "Fresnel Diffraction" in Towne
– – Section 13-1 "Introduction"
– – Section 13-2 "Fresnel Approximation for the radiation pattern of a linear distribution of coherent sources"
– – Section 13-3 "The Fresnel integrals and the Cornu spiral"
– – Section 13-4 "The Fresnel diffraction pattern of a single slit"
– – Section 13-5 "Fresnel diffraction by a wide slit"