Physics 380H - Wave Theory | **Homework #03 - Solutions**

Fall 2004 | Due 12:01 PM, Monday 2004/10/04

[50 points total]

"Journal" questions:
– Considering your entire history as a student, what course (in any subject) has been the most rewarding for you? Why?
– Any comments about this week's activities? Course content? Assignment? Lab?

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1. (From Towne P2-6. pg 36) Show from the basic equations of the acoustic approximation that p, s, ξ, and $\dot{\xi}$ all satisfy the one-dimensional wave equation. [10]

Solution: Following the work by Towne in Section 2-2, the following three equations are our foundation where (1.1) is the conservation of mass, (1.2) is Newton's 2nd law, and (1.3) is the equation of state:

$$\text{Conservation of mass:} \qquad \rho(x,t) = \rho_0 \left[1 + \frac{\partial \xi}{\partial x}(x,t)\right]^{-1} \tag{1.1}$$

$$\text{Newton's 2nd law:} \qquad -\frac{\partial P}{\partial x}(x,t) = \rho_0 \frac{\partial^2 \xi}{\partial t^2}(x,t) \tag{1.2}$$

$$\text{Equation of state:} \qquad P = P(\rho). \tag{1.3}$$

Taking the derivative with respect to x of (1.1) and of (1.3) we get

$$\frac{\partial \rho}{\partial x} = -\rho_0 \left[1 + \frac{\partial \xi}{\partial x}\right]^{-2} \frac{\partial^2 \xi}{\partial x^2} \tag{1.4}$$

$$\frac{\partial P}{\partial x} = P'(\rho) \frac{\partial \rho}{\partial x}. \tag{1.5}$$

Putting (1.4) and (1.5) gives us

$$\frac{\partial P}{\partial x} = -P'(\rho)\rho_0 \left[1 + \frac{\partial \xi}{\partial x}\right]^{-2} \frac{\partial^2 \xi}{\partial x^2}. \tag{1.6}$$

Using (1.2) in (1.6) gives us the relation of (1.6) where ρ is a function of ξ as given in (1.1):

$$-\rho_0 \frac{\partial^2 \xi}{\partial t^2} = -P'(\rho)\rho_0 \left[1 + \frac{\partial \xi}{\partial x}\right]^{-2} \frac{\partial^2 \xi}{\partial x^2}$$

$$\frac{\partial^2 \xi}{\partial t^2} = P'(\rho) \left[1 + \frac{\partial \xi}{\partial x}\right]^{-2} \frac{\partial^2 \xi}{\partial x^2}. \tag{1.7}$$

The acoustic approximation in (1.1) and (1.7) then gives us the result:

$$\frac{\partial \xi}{\partial x} \ll 1 \quad \Longrightarrow \quad \rho(x,t) \approx \rho_0$$

$$\frac{\partial^2 \xi}{\partial t^2} \approx P'(\rho_0) \frac{\partial^2 \xi}{\partial x^2}$$

$$\frac{\partial^2 \xi}{\partial t^2} = c^2 \frac{\partial^2 \xi}{\partial x^2}. \tag{1.8}$$

We can see that (1.8) is the one dimensional wave equation with a wave velocity given by the constant $c^2 = P'(\rho_0)$, thus ξ satisfies the wave equation in the acoustic approximation. It follows that ξ is of the form

$$\xi(x,t) = f(x+ct) + g(x-ct). \tag{1.9}$$

For $\dot{\xi}$ we can take the time derivative of (1.9) to arrive at

$$\begin{aligned}
\dot{\xi}(x,t) &= \frac{\partial \xi}{\partial t}(x,t) = cf'(x+ct) - cg'(x-ct) \\
\dot{\xi}(x,t) &= F(x+ct) + G(x-ct), \qquad (1.10)
\end{aligned}$$

where F and G are new functions defined by the derivatives of f and g multiplied by constants. Thus $\dot{\xi}$ is also a solution to a wave equation, though different than (1.8), but with the same wave speed.

Similarly, ξ' is also the solution to a wave equation, shown by taking the derivative of (1.9) with respect to x,

$$\begin{aligned}
-\xi'(x,t) = -\frac{\partial \xi}{\partial x}(x,t) = -s(x,t) &= f'(x+ct) + g'(x-ct) \\
s(x,t) &= -f'(x+ct) - g'(x-ct), \qquad (1.11)
\end{aligned}$$

showing that s is also the solution to a further wave equation, again with the same wave speed.

Since $p = \mathcal{B}_a s$ is merely s multiplied by a constant, it too satisfies yet another wave equation, since it has the correct form of being a function of $(x+ct)$ and/or $(x-ct)$, so once again the speed of the wave is the same as the other cases.

2. (From Towne P2-10. pg 36) Consider a progressive sinusoidal plane wave of given frequency ω and displacement amplitude ξ_m travelling in air, and a wave having the same values of ω and ξ_m traveling in water. How do the pressure amplitudes compare? If the value of s_m for the wave in water is sufficiently small to satisfy the acoustic approximation, is it necessarily so for the wave in air?. [10]

Solution: For the two situations we have similar equations for the displacement ξ, namely:

$$\begin{aligned}
\xi_w(x,t) &= \xi_m \sin(\omega t - k_w x) && \text{wave in water} \qquad (2.1) \\
\xi_a(x,t) &= \xi_m \sin(\omega t - k_a x) && \text{wave in air,} \qquad (2.2)
\end{aligned}$$

where the speed of sound is given by $c_a = \omega/k_a$ and $c_w = \omega/k_w$.

The relationship between pressure amplitudes and displacement amplitudes is given by

$$p(x,t) = -\mathcal{B}_a \frac{\partial \xi}{\partial x}, \tag{2.3}$$

so applying (2.3) to (2.1) and (2.2) gives us

$$\begin{aligned}
\text{water:} \quad p_w(x,t) &= -\mathcal{B}_{aw}\xi_m k_w \cos(\omega t - k_w x) = p_{mw}\cos(\omega t - k_w x) \qquad (2.4) \\
\text{air:} \quad p_a(x,t) &= -\mathcal{B}_{aa}\xi_m k_a \cos(\omega t - k_a x) = p_{ma}\cos(\omega t - k_a x). \qquad (2.5)
\end{aligned}$$

Dividing the expressions for p_{mw} by p_{ma} from (2.3) and (2.3) gives us the relationship of interest, where we have to then find out values for the various physical parameters for air and water from Towne Appendix IV, noting that $\mathcal{B}_a = \rho_0 c^2$ and $k = \omega/c$,

$$\begin{aligned}
\frac{p_{mw}}{p_{ma}} &= \frac{\mathcal{B}_{aw}\xi_m k_w}{\mathcal{B}_{aa}\xi_m k_a} = \frac{\mathcal{B}_{aw}k_w}{\mathcal{B}_{aa}k_a} \\
&= \frac{\mathcal{B}_{aw}\omega c_a}{\mathcal{B}_{aa}\omega c_w} = \frac{\rho_{0w}c_w^2 c_a}{\rho_{0a}c_a^2 c_w} \\
&= \frac{\rho_{0w}c_w}{\rho_{0a}c_a} = \frac{(998)(1483)}{(1.293)(331)} \\
&= 3458.16\ldots \approx 3460.
\end{aligned}$$

Thus the pressure amplitude of the wave in water is about 3460 times that of the wave in air.

The acoustic approximation depends on the x derivative of ξ being small, and these derivatives are given by

$$\frac{\partial \xi_w}{\partial x} = -k_w \xi_m \cos(\omega t - k_w x) \tag{2.6}$$

$$\frac{\partial \xi_a}{\partial x} = -k_a \xi_m \cos(\omega t - k_a x). \tag{2.7}$$

If the approximation holds for water, that means that (2.6) is small, and $k_w \xi_m$ is also small. How does this compare with the values for air? We see that

$$\begin{aligned}
\frac{k_a \xi_m}{k_w \xi_m} &= \frac{k_a}{k_w} \\
&= \frac{\omega c_w}{\omega c_a} = \frac{c_w}{c_a} \\
&= \frac{(998)}{(331)} = 3.015\ldots \approx 3.
\end{aligned}$$

If the acoustic approximation holds for water, it does not necessarily hold for air, since for the same displacement amplitude and frequency, air has about three times the value for $\frac{\partial \xi_w}{\partial x}$.

3. (From Towne P2-20. pg 38) Determination of an acoustic wave from given initial conditions:

(a) For plane waves satisfying the linearized one-dimensional wave equation in a uniform fluid medium of infinite extent, assume that the function $\dot{\xi}(x,0)$, which specifies the initial velocities of all the particles, and the initial pressure distribution, $p(x,0)$, are given. Find the general expression for $p(x,t)$. [10]

Solution: Let us define some functions for the initial conditions, the constants used are done to make the later algebra a bit easier:

$$\dot{\xi}(x,0) = \frac{c}{\mathcal{B}_a}\chi(x) \quad\implies\quad \chi(x) = \frac{\mathcal{B}_a}{c}\dot{\xi}(x,0) = Z\dot{\xi}(x,0) \tag{3.01}$$

$$p(x,0) = \Psi(x), \tag{3.02}$$

and the general solutions for the wave equation for $p(x,t)$ and $\xi(x,t)$ and various derivatives are given by

$$p(x,t) = f(x-ct) + g(x+ct) \tag{3.03}$$

$$\frac{\partial p}{\partial t}(x,t) = -cf'(x-ct) + cg'(x+ct) \tag{3.04}$$

$$\xi(x,t) = F(x-ct) + G(x+ct) \tag{3.05}$$

$$\dot{\xi}(x,t) = \frac{\partial \xi}{\partial t}(x,t) = -cF'(x-ct) + cG'(x+ct) \tag{3.06}$$

$$\frac{\partial \xi}{\partial x}(x,t) = F'(x-ct) + G'(x+ct) \tag{3.07}$$

Putting (3.02) into (3.03) and (3.01) into (3.06) gives us

$$\Psi(x) = f(x) + g(x) \tag{3.08}$$

$$\frac{c}{\mathcal{B}_a}\chi(x) = -cF'(x) + cG'(x). \tag{3.09}$$

Since p is related to s and ξ by (3.10), we can use that to further advance our knowledge in combination with (3.03) and (3.07).

$$p(x,t) = \mathcal{B}_a s = -\mathcal{B}_a \frac{\partial \xi}{\partial x}(x,t) \tag{3.10}$$

$$f(x-ct) + g(x+ct) = -\mathcal{B}_a F'(x-ct) - \mathcal{B}_a G'(x+ct) \tag{3.11}$$

This must hold for all values of x and t so (3.11) gives us:

$$f(x-ct) = -\mathcal{B}_a F'(x-ct) \tag{3.12}$$

$$g(x+ct) = -\mathcal{B}_a G'(x+ct) \tag{3.13}$$

We can now rewrite (3.09) to eliminate F' and G' to arrive at

$$\frac{c}{\mathcal{B}_a}\chi(x) = \frac{c}{\mathcal{B}_a}f(x) - \frac{c}{\mathcal{B}_a}g(x)$$

$$\chi(x) = f(x) - g(x) \tag{3.14}$$

Taking the sum and difference of $\Psi(x)$ and and $\chi(x)$ gives us forms for f and g, namely:

$$\Psi(x) + \chi(x) = 2f(x) \quad\implies\quad f(x) = \frac{1}{2}\left[\Psi(x) + \chi(x)\right]$$

$$f(x-ct) = \frac{1}{2}\left[\Psi(x-ct) + \chi(x-ct)\right] \tag{3.15}$$

$$\Psi(x) - \chi(x) = 2g(x) \quad\implies\quad g(x) = \frac{1}{2}\left[\Psi(x) - \chi(x)\right]$$

$$g(x+ct) = \frac{1}{2}\left[\Psi(x+ct) - \chi(x+ct)\right] \tag{3.16}$$

We can now use these expressions for f and g in the general solution (3.03) as well as to (3.01) and (3.02) arrive at

$$\begin{aligned} p(x,t) &= \frac{1}{2}\left[\Psi(x-ct)+\chi(x-ct)\right]+\frac{1}{2}\left[\Psi(x+ct)-\chi(x+ct)\right] \\ &= \frac{1}{2}\left[\Psi(x-ct)+\Psi(x+ct)\right]+\frac{1}{2}\left[\chi(x-ct)-\chi(x+ct)\right] \\ &= \frac{1}{2}\left[p(x-ct,0)+p(x+ct,0)\right]+\frac{\mathcal{B}_a}{2c}\left[\dot{\xi}(x-ct,0)-\dot{\xi}(x+ct,0)\right] \\ &= \frac{1}{2}\left[p(x-ct,0)+p(x+ct,0)\right]+\frac{Z}{2}\left[\dot{\xi}(x-ct,0)-\dot{\xi}(x+ct,0)\right], \end{aligned} \tag{3.18}$$

where $Z = \mathcal{B}_a/c$.

(b) Apply your general result from part (a) to the special case of particles initially at rest,

$$p(x,0) = \begin{cases} p_0, & x < 0, \\ 0, & x \geq 0. \end{cases}$$

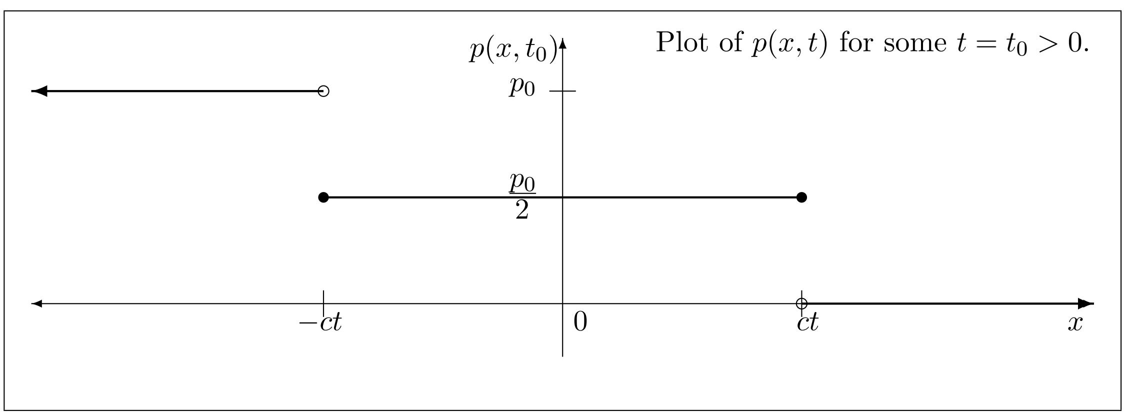
Plot or sketch the graph of $p(x,t)$ at some later time. (Such a situation would result if a long tube were divided in the middle by a membrane, the pressure on the two sides of the membrane being P_0 and P_0', and the membrane were ruptured at $t = 0$. For the linearized wave equation to apply, we must have $p_0 = P_0 - P_0' \ll P_0$.) [10]

Solution: Since the particles are initially at rest $\dot{\xi}(x,0) = 0$ and (3.18) reduces to

$$\begin{aligned} p(x,t) &= \frac{1}{2}\left[p(x-ct,0)+p(x+ct,0)\right]+\frac{Z}{2}\left[\dot{\xi}(x-ct,0)-\dot{\xi}(x+ct,0)\right] \\ &= \frac{1}{2}\left[p(x-ct,0)+p(x+ct,0)\right] \end{aligned} \tag{3.19}$$

We have left and right moving waves given by (3.19). Combining this information with the initial condition of $p(x,0)$ gives us the full form of the solution:

$$p(x,t) = \begin{cases} p_0, & -\infty < x < -ct, \\ \frac{1}{2}p_0, & -ct \leq x \leq +ct, \\ 0, & +ct < x < +\infty. \end{cases}$$



(c) Given that the initial pressure and velocity distributions satisfy the condition characteristic of a +wave, $p(x,0) = \rho_0 c\dot{\xi}(x,0)$, show that your general result from part (a) reduces to the pure +wave $p(x,t) = p(x-ct,0)$. [10]

Solution: If the initial pressure and velocity distributions satisfy the condition characteristic of a +wave, $p(x,0) = \rho_0 c\dot{\xi}(x,0)$, then (3.01) and (3.02) give us

$$\begin{aligned} p(x,0) &= \Psi(x) = \rho_0 c\dot{\xi}(x,0) = Z\dot{\xi}(x,0) = \chi(x) \\ \therefore \Psi(x) &= \chi(x), \end{aligned} \tag{3.20}$$

and (3.18) reduces to

$$\begin{aligned} p(x,t) &= \frac{1}{2}\left[\Psi(x-ct) + \Psi(x+ct)\right] + \frac{1}{2}\left[\chi(x-ct) - \chi(x+ct)\right] \\ &= \frac{1}{2}\left[\Psi(x-ct) + \Psi(x+ct)\right] + \frac{1}{2}\left[\Psi(x-ct) - \Psi(x+ct)\right] \\ &= \frac{1}{2}\left[\Psi(x-ct) + \Psi(x+ct) + \Psi(x-ct) - \Psi(x+ct)\right] \\ &= \frac{1}{2}\left[\Psi(x-ct) + \Psi(x-ct)\right] \\ &= \Psi(x-ct). \end{aligned} \tag{3.21}$$

Reapplying (3.02) we get the desired result

$$\begin{aligned} p(x,t) &= \Psi(x-ct) \\ &= p(x-ct,0). \end{aligned} \tag{3.22}$$

Headstart for next week, Week 04, starting Monday 2004/10/04:
– Read Chapter 3 “Boundary value problems” in “Wave Phenomena” by Towne, omit 3-9
– – Section 3-4 “Reflection and transmission at an interface”
– – Section 3-5 “Reflection of a sinusoidal wave; partial standing wave”
– – Section 3-6 “Extreme mismatch of impedances; rigid and free surfaces”
– – Section 3-7 “Reflection of a sinusoidal wave from a pair of interfaces”
– – Section 3-8 “Reflection of a sinusoidal wave at a pair of interfaces, alternate method”
– Read Chapter 4 “Energy in a Sound Wave; Isomorphisms” in Towne, omit 4-7
– – Section 4-1 “Energy density and energy flux for a plane sound wave”
– – Section 4-2 “The law of conservation of energy”
– – Section 4-3 “Separability of energy into + and – components”
– – Section 4-4 “Convective and radiative energy terms”
– – Section 4-5 “Relative radiative intensities in reflection and transmission at a single interface”
– – Section 4-6 “Intensity relations for progressive sinusoidal waves”