Physics 202H - Introductory Quantum Physics I
Homework #12 - Solutions

Fall 2004 Due 5:01 PM, Monday 2004/12/13

[70 points total]

"Journal" questions. Briefly share your thoughts on the following questions:
– What aspects of this course do you think you are most likely to use in the future, both in your "physics" existence and in your "day-to-day" life?
– Any comments about this week's activities? Course content? Assignment? Lab?

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1. Please complete the anonymous end of course survey online on WebCT. Constructive feedback will hopefully allow us to have the best possible courses in the future, and provide the instructor and department with useful information about student reactions to many aspects of the program. In addition to the bonus assignment marks, survey participation may count towards overall class participation scores. [5.01-bonus]

   Solution: Do the survey - get the bonus marks.

2. (From Eisberg & Resnick, Q 5-28 and 5-29, pg 169) Why is ψ necessarily an oscillatory function if $V(x) < E$? Why does ψ tend to go to infinity if $V(x) > E$? Limit your discussion to about 50 words or so. [10]

   Solution: The eigenfunction ψ is a solution to the time-independent Schroedinger equation, and thus depending on the sign of $V(x) - R$ will have a solution of either positive and negative exponentials (when $V(x) > E$) or positive and negative *imaginary* exponentials. Exponentials go to infinity for large values of x, and of course negative exponentials go to infinity for large negative values of x, compared with imaginary exponentials, which are oscillatory.

3. (From Eisberg & Resnick, P 5-4, pg 169) By evaluating the classical normalization integral in Example 5-6, show that the value of the constant $B^2 = \sqrt{(C/m\pi^2)}$ which satisfies the requirement that the total probability of finding the particle in the classical oscillator somewhere between its limits of motion must equal one. [10]

   Solution: From Eisberg & Resnick, Example 5-6, pg 136, we can arrive at an expression for the probability density P based on the classical calculation of the fraction of the time the oscillator is at each section of the range. This probability is inversely proportional to its velocity.

$$\begin{aligned} v &= \sqrt{\frac{2}{m}}\sqrt{E - \frac{Cx^2}{2}} \\ P &= \frac{B^2}{v} \\ &= \frac{B^2}{\sqrt{\frac{2}{m}}\sqrt{E - \frac{Cx^2}{2}}}. \end{aligned}$$

To find the value of B, we need to normalize this probability, insuring that the total probability of finding the particle in all allowed locations is unity.

$$\begin{aligned} 1 &= \int_{-\infty}^{+\infty} P \, \mathrm{d}x \\ &= \int_{-\infty}^{+\infty} \frac{B^2}{\sqrt{\frac{2}{m}}\sqrt{E - \frac{Cx^2}{2}}} \, \mathrm{d}x \\ &= \frac{B^2}{\sqrt{\frac{2}{m}}} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{E - \frac{Cx^2}{2}}} \, \mathrm{d}x \end{aligned}$$

E, the total energy, is a constant, and can be stated in terms of a, the maximum value for x, since at this point, the velocity of the particle will be zero.

$$\begin{aligned} E &= K + V \\ &= \frac{1}{2}mv^2 + \frac{1}{2}Cx^2 \\ &= \frac{1}{2}Ca^2 \end{aligned}$$

This gives us the integral that we need to evaluate, namely

$$\begin{aligned} 1 &= \frac{B^2}{\sqrt{\frac{2}{m}}} \int_{-a}^{+a} \frac{1}{\sqrt{\frac{Ca^2}{2} - \frac{Cx^2}{2}}} \, \mathrm{d}x \\ &= \frac{B^2}{\sqrt{\frac{2}{m}}} \int_{-a}^{+a} \frac{1}{\sqrt{\frac{C}{2}}\sqrt{a^2 - x^2}} \, \mathrm{d}x \\ &= B^2\sqrt{\frac{m}{C}} \int_{-a}^{+a} \frac{1}{\sqrt{a^2 - x^2}} \, \mathrm{d}x. \end{aligned}$$

Either looking up the integral, or doing a trigonometric substitution gives us the result that the integral is equal to $\arcsin(x/a)$

$$\begin{aligned} 1 &= B^2\sqrt{\frac{m}{C}} \int_{-a}^{+a} \frac{1}{\sqrt{a^2 - x^2}} \, \mathrm{d}x \\ &= B^2\sqrt{\frac{m}{C}} \left[\arcsin\left(\frac{x}{a}\right)\right]_{-a}^{+a} \\ &= B^2\sqrt{\frac{m}{C}} \left[\arcsin\left(\frac{a}{a}\right) - \arcsin\left(\frac{-a}{a}\right)\right] \\ &= B^2\sqrt{\frac{m}{C}} \left[\arcsin(1) - \arcsin(-1)\right]. \end{aligned}$$

The angle who's sin is 1 is $\pi/2$, and the angle who's sin is -1 is $-\pi/2$, so this expression becomes

$$\begin{aligned} 1 &= B^2\sqrt{\frac{m}{C}} \left[\arcsin(1) - \arcsin(-1)\right] \\ 1 &= B^2\sqrt{\frac{m}{C}} \left[\frac{\pi}{2} - \frac{-\pi}{2}\right] = B^2\pi\sqrt{\frac{m}{C}}. \end{aligned}$$

Solving for B^2 gives us

$$1 = B^2\pi\sqrt{\frac{m}{C}} \quad \Longrightarrow \quad B^2 = \frac{1}{\pi\sqrt{\frac{m}{C}}} = \sqrt{\frac{C}{m\pi^2}}.$$

4. (From Eisberg & Resnick, P 5-7, pg 170)

   (a) Use the particle in a box wave function verified in Example 5-9, with the value of $A^2 = 2/a$ determined in Example 5-10, to calculate the probability that the particle associated with the wave function would be found in a measurement within a distance of $a/3$ from the right-hand end of the box of length a. The particle is in its lowest energy state. [10]

   Solution: The probability of finding the particle between points x_1 and x_2 is given by

$$\begin{aligned}\int_{x_1}^{x_2} P(x,t)\,\mathrm{d}x &= \int_{x_1}^{x_2} \Psi^*(x,t)\Psi(x,t)\,\mathrm{d}x \\ &= \int_{x_1}^{x_2} \left(A\cos\left(\frac{\pi x}{a}\right)\mathrm{e}^{+\mathrm{i}Et/\hbar}\right)\left(A\cos\left(\frac{\pi x}{a}\right)\mathrm{e}^{-\mathrm{i}Et/\hbar}\right)\mathrm{d}x \\ &= \int_{x_1}^{x_2} A^2\cos^2\left(\frac{\pi x}{a}\right)\mathrm{d}x \\ &= A^2\left[\frac{x}{2} + \frac{\sin\left(2\pi x/a\right)}{4\pi/a}\right]_{x_1}^{x_2}.\end{aligned}$$

   For the situation described, $x_1 = a/2 - a/3 = a/6$ and $x_2 = a/2$, so we have

$$\begin{aligned}\int_{-x_1}^{x_2} P(x,t)\,\mathrm{d}x &= A^2\left[\frac{x}{2} + \frac{\sin\left(2\pi x/a\right)}{4\pi/a}\right]_{x_1}^{x_2} \\ &= \frac{2}{a}\left[\frac{x}{2} + \frac{\sin\left(2\pi x/a\right)}{4\pi/a}\right]_{a/6}^{a/2} \\ &= \left[\frac{x}{a} + \frac{\sin\left(2\pi x/a\right)}{2\pi}\right]_{a/6}^{a/2} \\ &= \left[\frac{1}{2} + \frac{\sin\left(\pi\right)}{2\pi} - \frac{1}{6} - \frac{\sin\left(\pi/3\right)}{2\pi}\right] \\ &= \frac{1}{2} + 0 - \frac{1}{6} - \frac{\sqrt{3}/2}{2\pi} \\ &= \frac{1}{3} - \frac{\sqrt{3}}{4\pi} \approx 0.1955\end{aligned}$$

   The probability that the particle associated with the wave function would be found in a measurement within a distance of $a/3$ from the right-hand end of the box of length a is about 0.1955, or just under 20%.

   (b) Compare with the probability that would be predicted classically from a simple calculation related to the one in Example 5-6, for a particle with constant speed bouncing back and forth from the ends of the box. [5]

   Solution: For a particle with constant speed, the probability of finding it at any point in the box is equal, and the probability density is just $1/a$. Thus,

$$\begin{aligned}\int_{x_1}^{x_2} P(x,t)\,\mathrm{d}x &= \frac{1}{a}(x_1 - x_2) \\ &= \frac{1}{2} - \frac{1}{6} = \frac{1}{3} \approx 0.3333.\end{aligned}$$

   For a classical particle with constant speed, the probability that the particle would be found in a measurement within a distance of $a/3$ from the right-hand end of the box of length a is 1/3 or a bit more than 33%

(c) Compare with the probability that would be predicted classically from a calculation related to the one in Example 5-6, for a particle undergoing simple harmonic motion with the value of $B^2 = \sqrt{(C/m\pi^2)}$. [5]

Solution: Building on the calculations carried out in the previous problem, with the small change that the previous problem had a box size of $2a$ and this one has a size of a, the probability of finding the particle between points x_1 and x_2 for a particle undergoing simple harmonic motion is given by

$$\begin{aligned}
\int_{x_1}^{x_2} P(x,t)\,\mathrm{d}x &= \sqrt{\frac{C}{m\pi^2}}\sqrt{\frac{m}{C}}\int_{x_1}^{x_2} \frac{1}{\sqrt{(a/2)^2 - x^2}}\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_{a/6}^{a/2} \frac{1}{\sqrt{(a/2)^2 - x^2}}\,\mathrm{d}x \\
&= \frac{1}{\pi}\left[\arcsin\left(\frac{2x}{a}\right)\right]_{a/6}^{a/2} \\
&= \frac{1}{\pi}\left[\arcsin(1) - \arcsin\left(\frac{1}{3}\right)\right] \\
&= \frac{1}{\pi}\left[\frac{\pi}{2} - \arcsin\left(\frac{1}{3}\right)\right] \\
&= \frac{1}{2} - \frac{\arcsin(1/3)}{\pi} \approx 0.3918
\end{aligned}$$

For a classical particle undergoing simple harmonic motion , the probability that the particle would be found in a measurement within a distance of $a/3$ from the right-hand end of the box of length a is about 0.3918 or a bit less than 40%.

5. (From Eisberg & Resnick, P 6-11 and 6-12, pg 229)

(a) Verify by substitution that the standing wave general solution, (Eisberg & Resnick, Equation 6-62, pg 211), satisfies the time-independent Schroedinger equation, (Eisberg & Resnick, Equation 6-2, pg 178), for the finite square well potential in the region inside the well. [5]

Solution: The time-independant Schroedinger equation is

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\psi(x)}{\mathrm{d}x^2} + V(x,t)\psi(x) = E\psi(x). \tag{5.1}$$

For the region inside the well, the potential $V(x) = 0$, so the time-independent Schroedinger equation becomes

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\psi(x)}{\mathrm{d}x^2} = E\psi(x). \tag{5.2}$$

The standing wave general solution is

$$\psi_a(x) = A'\sin k_I x + B'\cos k_I x \quad \text{where} \quad k_I = \frac{\sqrt{2mE}}{\hbar}, \qquad -\frac{a}{2} < x < +\frac{a}{2}. \tag{5.3}$$

The derivatives of $\psi_a(x)$ are:

$$\begin{aligned} \frac{\mathrm{d}\psi_a(x)}{\mathrm{d}x} &= k_I A'\cos k_I x - k_I B'\sin k_I x \\ \frac{\mathrm{d}^2\psi_a(x)}{\mathrm{d}x^2} &= -k_I^2 A'\sin k_I x - k_I^2 B'\cos k_I x \\ &= -k_I^2\left(A'\sin k_I x + B'\cos k_I x\right) \\ &= -k_I^2\psi_a(x) \\ &= -\frac{2mE}{\hbar^2}\psi_a(x). \end{aligned} \tag{5.4}$$

If we multiply (5.4) by $-\hbar^2/2m$ we get

$$\left(-\frac{\hbar^2}{2m}\right)\frac{\mathrm{d}^2\psi_a(x)}{\mathrm{d}x^2} = \left(-\frac{\hbar^2}{2m}\right)\left(-\frac{2mE}{\hbar^2}\right)\psi_a(x) = E\psi_a(x),$$

which is (5.2), the time-independent Schroedinger equation in the region inside the well. Thus we have shown that (5.3) is a solution to the time-independent Schroedinger equation (5.1) in the specified region.

(b) Verify by substitution that the exponential general solution, (Eisberg & Resnick, Equation 6-63 and 6-64, pg 212), satisfy the time-independent Schroedinger equation, (Eisberg & Resnick, Equation 6-13, pg 186), for the finite square well potential in the regions outside the well. [5]

Solution: For the region outside the well, the potential $V(x) = V_0$, so the time-independent Schroedinger equation becomes

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\psi(x)}{\mathrm{d}x^2} + V_0\psi(x) = E\psi(x). \tag{5.5}$$

The exponential general solutions are

$$\psi_b(x) = C\mathrm{e}^{k_{II}x} + D\mathrm{e}^{-k_{II}x} \quad \text{where} \quad k_{II} = \frac{\sqrt{2m(V_0 - E)}}{\hbar}, \qquad x < -\frac{a}{2}, \tag{5.6}$$

and

$$\psi_c(x) = F\mathrm{e}^{k_{II}x} + G\mathrm{e}^{-k_{II}x} \quad \text{where} \quad k_{II} = \frac{\sqrt{2m(V_0 - E)}}{\hbar}, \qquad x > +\frac{a}{2}. \tag{5.7}$$

The derivatives of $\psi_b(x)$ are:

$$\begin{aligned}
\frac{\mathrm{d}\psi_b(x)}{\mathrm{d}x} &= k_{II}C\mathrm{e}^{k_{II}x} - k_{II}D\mathrm{e}^{-k_{II}x} \\
\frac{\mathrm{d}^2\psi_b(x)}{\mathrm{d}x^2} &= k_{II}^2C\mathrm{e}^{k_{II}x} + k_{II}^2D\mathrm{e}^{-k_{II}x} \\
&= k_{II}^2\left(C\mathrm{e}^{k_{II}x} + D\mathrm{e}^{-k_{II}x}\right) \\
&= k_{II}^2\psi_b(x) \\
&= \frac{2m(V_0 - E)}{\hbar^2}\psi_c(x).
\end{aligned} \tag{5.8}$$

If we multiply (5.8) by $-\hbar^2/2m$ and add $V_0\psi_b(x)$ we get

$$\left(-\frac{\hbar^2}{2m}\right)\frac{\mathrm{d}^2\psi_b(x)}{\mathrm{d}x^2} + V_0\psi_b(x) = \left(-\frac{\hbar^2}{2m}\right)\left(-\frac{2m(V_0 - E)}{\hbar^2}\right)\psi_b(x) + V_0\psi_b(x) = E\psi_b(x),$$

which is (5.5), the time-independent Schroedinger equation in the region outside the well. Thus we have shown that (5.6) is a solution to the time-independent Schroedinger equation (5.1) in the specified region.

Similarly, for $\psi_c(x)$ we have:

$$\begin{aligned}
\frac{\mathrm{d}^2\psi_c(x)}{\mathrm{d}x^2} &= k_{II}^2\psi_c(x) \\
&= \frac{2m(V_0 - E)}{\hbar^2}\psi_c(x),
\end{aligned} \tag{5.9}$$

and if we multiply (5.9) by $-\hbar^2/2m$ and add $V_0\psi_c(x)$ we get

$$\left(-\frac{\hbar^2}{2m}\right)\frac{\mathrm{d}^2\psi_c(x)}{\mathrm{d}x^2} + V_0\psi_c(x) = \left(-\frac{\hbar^2}{2m}\right)\left(-\frac{2m(V_0 - E)}{\hbar^2}\right)\psi_c(x) + V_0\psi_c(x) = E\psi_c(x),$$

which is (5.5), the time-independent Schroedinger equation in the region outside the well. Thus we have shown that (5.7) is a solution to the time-independent Schroedinger equation (5.1) in the specified region.

6. (From Eisberg & Resnick, Q 6-15, pg 227) A particle is incident on a potential barrier of width a, with total energy less than the barrier height, and it is reflected. Does the reflection involve only the potential discontinuity facing its direction of incidence? If the other discontinuity were moved by increasing a, is the reflection coefficient changed? What if the other discontinuity were removed, so that the barrier was changed into a step? Limit your discussion to about 50 words or so. [10]

   Solution: The reflection does *not* only involve the potential discontinuity facing the direction of incidence. The width of the barrier has an effect on the reflection and transmission coefficients, as the barrier width is increased the amount of reflection in general decreases (thought there are some particular widths and energies for which the amount of reflection is particularly high or low due to constructive or destructive interference between the waves reflected off of each discontinuity.) If the second discontinuity were removed, changing the barrier into a step, then there would be 100% reflection, compared to smaller reflections for barriers of finite thickness.

7. (From Eisberg & Resnick, Q 6-34, pg 231) Verify the eigenfunction and eigenvalue for the $n = 2$ state of a simple harmonic oscillator by direct substitution into the time-independent Schroedinger equation, as in (Eisberg & Resnick, Example 6-7, pg 224). [10]

   Solution: The time-independant Schroedinger equation is

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\psi(x)}{\mathrm{d}x^2} + V(x,t)\psi(x) = E\psi(x). \tag{7.1}$$

For the simple harmonic oscillator, the potential $V(x) = Cx^2/2$, so the time-independent Schroedinger equation becomes

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\psi(x)}{\mathrm{d}x^2} + \frac{C}{2}x^2\psi(x) = E\psi(x).$$

For $u^2 = x^2\sqrt{Cm}/\hbar$, and $E_2 = \hbar(5/2)\sqrt{C/m}$, this gives us

$$\begin{aligned}\frac{\mathrm{d}^2\psi_2}{\mathrm{d}x^2} &= \frac{2m}{\hbar^2}\frac{C}{2}x^2\psi_2 - \frac{2m}{\hbar^2}E_2\psi_2 = \frac{2m}{\hbar^2}\frac{C}{2}\frac{\hbar}{\sqrt{Cm}}u^2\psi_2 - \frac{2m}{\hbar^2}\hbar\frac{5}{2}\sqrt{\frac{C}{m}}\psi_2 \\ &= \frac{\sqrt{Cm}}{\hbar}u^2\psi_2 - \frac{5\sqrt{Cm}}{\hbar}\psi_2 = \left(u^2 - 5\right)\frac{\sqrt{Cm}}{\hbar}\psi_2.\end{aligned} \tag{7.2}$$

The $n = 2$ eigenfunction is

$$\psi_2(u) = A_2\left(1 - 2u^2\right)\mathrm{e}^{-u^2/2} \quad \text{where} \quad u = \frac{(Cm)^{1/4}}{\hbar^{1/2}}x. \tag{7.3}$$

The first derivative of $\psi_2(x)$ is:

$$\begin{aligned}\frac{\mathrm{d}\psi_2(x)}{\mathrm{d}x} &= \frac{\mathrm{d}\psi_2(u)}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}\left(A_2\left(1-2u^2\right)\mathrm{e}^{-u^2/2}\right)}{\mathrm{d}u}\frac{\mathrm{d}\left(\frac{(Cm)^{1/4}}{\hbar^{1/2}}x\right)}{\mathrm{d}x} \\ &= A_2\left(\frac{\mathrm{d}\left(1-2u^2\right)}{\mathrm{d}u}\mathrm{e}^{-u^2/2} + \left(1-2u^2\right)\frac{\mathrm{d}\left(\mathrm{e}^{-u^2/2}\right)}{\mathrm{d}u}\right)\left(\frac{(Cm)^{1/4}}{\hbar^{1/2}}\right) \\ &= A_2\left(-4u\mathrm{e}^{-u^2/2} + \left(1-2u^2\right)\left(\frac{-2u}{2}\right)\mathrm{e}^{-u^2/2}\right)\left(\frac{(Cm)^{1/4}}{\hbar^{1/2}}\right) \\ &= u\left(-4A_2\mathrm{e}^{-u^2/2} - A_2\left(1-2u^2\right)\mathrm{e}^{-u^2/2}\right)\left(\frac{(Cm)^{1/4}}{\hbar^{1/2}}\right) \\ &= u\left(-4A_2\mathrm{e}^{-u^2/2} - \psi_2\right)\left(\frac{(Cm)^{1/4}}{\hbar^{1/2}}\right) = \left(-4A_2u\mathrm{e}^{-u^2/2} - u\psi_2\right)\left(\frac{(Cm)^{1/4}}{\hbar^{1/2}}\right).\end{aligned}$$

The second derivative of ψ is:

$$
\begin{aligned}
\frac{\mathrm{d}^2\psi_2(x)}{\mathrm{d}x^2} &= \frac{\mathrm{d}\frac{\mathrm{d}\psi_2(x)}{\mathrm{d}x}}{\mathrm{d}x} = \frac{\mathrm{d}\frac{\mathrm{d}\psi_2(u)}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x} \\
&= \frac{\mathrm{d}^2\psi_2(u)}{\mathrm{d}u^2}\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2 \\
&= \frac{\mathrm{d}\left(-4A_2 u\mathrm{e}^{-u^2/2} - u\psi_2\right)}{\mathrm{d}u}\left(\frac{(Cm)^{1/4}}{\hbar^{1/2}}\right)^2 \\
&= -\left(\frac{\mathrm{d}\left(4A_2 u\mathrm{e}^{-u^2/2}\right)}{\mathrm{d}u} + \frac{\mathrm{d}\left(u\psi_2\right)}{\mathrm{d}u}\right)\frac{\sqrt{Cm}}{\hbar} \\
&= -\left(\frac{\mathrm{d}\left(4A_2 u\right)}{\mathrm{d}u}\mathrm{e}^{-u^2/2} + 4A_2 u\frac{\mathrm{d}\left(\mathrm{e}^{-u^2/2}\right)}{\mathrm{d}u} + \frac{\mathrm{d}\left(u\right)}{\mathrm{d}u}\psi_2 + u\frac{\mathrm{d}\left(\psi_2\right)}{\mathrm{d}u}\right)\frac{\sqrt{Cm}}{\hbar} \\
&= -\left(4A_2\mathrm{e}^{-u^2/2} - 4A_2u^2\mathrm{e}^{-u^2/2} + \psi_2 + u\left(-4A_2u\mathrm{e}^{-u^2/2} - u\psi_2\right)\right)\frac{\sqrt{Cm}}{\hbar} \\
&= -\left(4A_2\mathrm{e}^{-u^2/2} - 4A_2u^2\mathrm{e}^{-u^2/2} + \psi_2 - 4A_2u^2\mathrm{e}^{-u^2/2} - u^2\psi_2\right)\frac{\sqrt{Cm}}{\hbar} \\
&= \left(-4A_2\mathrm{e}^{-u^2/2} + 8A_2u^2\mathrm{e}^{-u^2/2} - \psi_2 + u^2\psi_2\right)\frac{\sqrt{Cm}}{\hbar} \\
&= \left(-4A_2\left(1-2u^2\right)\mathrm{e}^{-u^2/2} - \psi_2 + u^2\psi_2\right)\frac{\sqrt{Cm}}{\hbar} \\
&= \left(-4\psi_2 - \psi_2 + u^2\psi_2\right)\frac{\sqrt{Cm}}{\hbar} \\
&= \left(u^2 - 5\right)\frac{\sqrt{Cm}}{\hbar}\psi_2,
\end{aligned}
$$

which is (7.2), the time-independent Schroedinger equation for the harmonic potential. Thus we have shown that (7.3) is a solution to the time-independent Schroedinger equation (7.1) for the specified potential.

Headstart for next week, Week 13, starting Monday 2004/12/13:
– Review notes, review texts, review assignments, learn material, do well on exam