Physics 202H - Introductory Quantum Physics I
Homework #02 - Solutions

Fall 2004 Due 5:01 PM, Monday 2004/09/27

[65 points total]

"Journal" questions. Briefly share your thoughts on the following questions:
– What is your major/minor/etc.? What are you planning on doing after you finish your degree?
– Any comments about this week's activities? Course content? Assignment? Lab?

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1. (From Eisberg & Resnick, Q 1-8, pg 22) In your own words give a brief (not more than 50 words) explanation of the "ultraviolet catastrophe". [10]

   Solution: See page 12 and 13 of Eisberg & Resnick.

   The discrepancy between the classical model of cavity radiation and the experimental facts was one of the big mysteries of physics at the turn of the century. The classical calculations seemed to indicate that at any temperature, a black body should emit an infinite amount of energy at very high frequencies (very short wavelengths), which of course does not happen. The word "ultraviolet" is in reference to these high frequencies. I think the word "catastrophe" is in reference to the idea of large amounts of energy being emitted, like having a bomb go off, rather than the intellectual/academic/scientific "catastrophe" of not having a theory that agrees with experiment.

2. A pendulum of length $l = 0.1\,\mathrm{m}$ and mass $m = 0.01\,\mathrm{kg}$ swings up to a maximum angle of $\theta = 0.1\,\mathrm{rad}$. If its energy is quantized, the discontinuous jumps in energy are very small.

   (a) For this angle of swing, what is the quantum number n that corresponds with the total kinetic energy of the system? What does this mean? [5]

   Solution: We first need to find the energy and frequency of the pendulum. To find the energy we need to apply some geometry to the situation to figure out the height h that the pendulum is lifted to. Constructing a triangle, with hypotenuse of length l and height of length $(l - h)$ we get the relationship:

   $$\cos\theta = fracl - hl \quad \Longrightarrow \quad h = l - l\cos\theta = l\,(1 - \cos\theta)\,.$$

   Given the acceleration of gravity $g \approx 9.80665\,\mathrm{m/s^2}$, the total (classical) energy of the swinging pendulum and the frequency ν of oscillation is thus

   $$E_c = mgh = mgl\,(1 - \cos\theta) \quad \text{and} \quad \nu = \frac{1}{2\pi}\sqrt{\frac{g}{l}}.$$

   If this system were to be described by quantum physics, the energy must satisfy the relation

   $$\mathcal{E} = nh\nu \quad n = 0, 1, 2, 3, \ldots$$

   where $h = 6.6260755 \times 10^{34}\,\mathrm{Js}$ is Planck's constant. So if we set $E_c = \mathcal{E}$ we get

   $$mgl\,(1 - \cos\theta) = nh\nu = nh\frac{1}{2\pi}\sqrt{\frac{g}{l}}$$

and solving for n gives us

$$\begin{aligned} n &= \frac{mgl}{h\nu}\left(1-\cos\theta\right) \\ &= \frac{2\pi\ mgl}{h}\sqrt{\frac{l}{g}}\left(1-\cos\theta\right) \\ &= \frac{2\pi\ m}{h}\sqrt{gl^3}\left(1-\cos\theta\right) \\ &\approx 4.6912857628\times 10^{28} \end{aligned}$$

which is clearly a HUGE number. If you look at Example 1-6 from Eisberg & Resnick, page 21, we see that adjacent energy levels for this system are only about $\Delta E \approx 10^{-33}$ joules apart, which fits with our calculation above. Of course variations for our values for m, g, l, θ, etc. make it impossible to confirm the discrete quantum nature of this type of system. According to quantum theory, n must be an integer, but of course for this calculation, there is no way that our physical measurements of the of the system could be accurate enough to determine if the calculated value for n is an integer or not.

(b) From a relativistic point of view, how much larger would the measured mass of the swinging system be than the rest mass of the stationary system? What does this mean? [5]

Solution: The extra energy of the swinging, E_c calculated above would appear as extra mass were the whole swinging system be measured on a sensitive enough scale. While one can consider the rest mass of a particle to be a relativistic invariant, the measured mass of a system of particles (such as an atom or molecule or something more complex) is not just the sum of the rest masses of the component parts of that system – this total mass is also effected by the energies of all of the components, kinetic as well as potential energies. Thus a water molecule has a mass that is less than the sum of the masses of two hydrogen atoms and one oxygen atom, and the mass of a compressed spring is larger (by a tiny tiny amount) than the mass of an uncompressed spring.

For this example, the total relativistic energy of the system is given by

$$E = E_c + m_0c^2 = m_rc^2$$

and this energy is equivalent to a mass of

$$m_r = (E_c + m_0c^2)\frac{1}{c^2} = \frac{E_c}{c^2} + m_0$$

The change in measured mass would be

$$\Delta m_r = m_r - m_0 = \frac{E_c}{c^2}.$$

If we assume that the rest mass of the system is equal to the only mass that we are given, m, (which assumes that the string is massless) then the actual numbers would be

$$\Delta m_r = \frac{E_c}{c^2} = \frac{mgl}{c^2}\left(1-\cos\theta\right) \approx 5.45\times 10^{-22}\text{kg}$$

which is a large mass when compared to atomic masses, but is a very tiny one to be measuring in the real world of grams and kilograms.

3. The peak of the radiation curve for a certain blackbody occurs at a wavelength of $\lambda_a = 1\,\mu$m. If the temperature is raised so that the total radiated power is increased 16–fold, at what wavelength λ_b will the new intensity maximum be found? [10]

   Solution: The wavelength of the intensity maximum is inversely related to the temperature by Wein's law $\lambda T =$constant, while the total radiated power is related to the temperature to the fourth power by Stefan's law $R_T = \sigma T^4$. Thus if the power goes up by a factor of 16, the temperature must have increased by a factor of 2, since $2^4 = 16$. If the temperature doubled, then the wavelength of the peak of the radiation curve must have halved, so $\lambda_b = \lambda_a/2 = 0.5\,\mu$m

4. (From Eisberg & Resnick, P 1-20, pg 24) Show that, at the wavelength $\lambda_{\max}$, where $\rho_T(\lambda)$ has its maximum, $\rho_T(\lambda_{\max}) \approx 170\pi\,(kT)^5/(hc)^4$. [10]

   Solution: We can take Planck's result in terms of the wavelength

$$\rho_T(\lambda)\mathrm{d}\lambda = \frac{8\pi hc}{\lambda^5}\frac{\mathrm{d}\lambda}{\left(\mathrm{e}^{hc/\lambda kT} - 1\right)}$$

and ignoring the factors of dλ (by "dividing" them out of the way for example),

$$\rho_T(\lambda) = \frac{8\pi hc}{\lambda^5}\frac{1}{\left(\mathrm{e}^{hc/\lambda kT} - 1\right)} \tag{1}$$

we need only apply Wien's displacement law (2)

$$\lambda_{max} T = \text{constant} = 0.2014\frac{hc}{k} \quad\Longrightarrow\quad \lambda_{max} = 0.2014\frac{hc}{kT} \tag{2}$$

to (1) to give us

$$\begin{aligned}
\rho_T(\lambda_{max}) &= \frac{8\pi hc}{\lambda_{max}^5}\frac{1}{\left(\mathrm{e}^{hc/\lambda_{max}kT} - 1\right)} \\
&= \frac{8\pi hc}{\left(0.2014\frac{hc}{kT}\right)^5}\frac{1}{\left(\mathrm{e}^{hc/\left(0.2014\frac{hc}{kT}\right)kT} - 1\right)} \\
&= \frac{8\pi (kT)^5}{(0.2014)^5 (hc)^4}\frac{1}{\left(\mathrm{e}^{1/0.2014} - 1\right)} \\
&= \frac{8}{(0.2014)^5}\frac{1}{\left(\mathrm{e}^{1/0.2014} - 1\right)}\pi\frac{(kT)^5}{(hc)^4} \\
&= \frac{8}{(0.2014)^5}\frac{1}{\left(\mathrm{e}^{4.96524\ldots} - 1\right)}\pi\frac{(kT)^5}{(hc)^4} \\
&= (24143.079\ldots)\frac{1}{(143.34\ldots - 1)}\pi\frac{(kT)^5}{(hc)^4} \\
&= (24143.079\ldots)\frac{1}{(142.34\ldots)}\pi\frac{(kT)^5}{(hc)^4} \\
&= (24143.079\ldots)(0.00702\ldots)\pi\frac{(kT)^5}{(hc)^4} \\
&= (169.61\ldots)\pi\frac{(kT)^5}{(hc)^4} \\
\therefore\quad \rho_T(\lambda_{max}) &\approx 170\pi\frac{(kT)^5}{(hc)^4}.
\end{aligned}$$

5. Think of the Sun as a blackbody radiator at temperature $T = 5777\,\mathrm{K}$.

   (a) How much power is radiated by each square metre of the Sun's surface? [5]

   Solution: Stefan's law with the Stefan-Boltzman constant gives us a way of calculating the radiancy, or the power radiated per unit area, namely:

$$
\begin{aligned}
R_T &= \sigma T^4 \quad \text{where} \quad \sigma = 5.670400 \times 10^{-8}\,\mathrm{W/m^{2\circ}K^4} \\
R_{T_\odot} &= \sigma T_\odot^4 \\
R_{T_\odot} &= (5.670400 \times 10^{-8}\,\mathrm{W/m^{2\circ}K^4})(5777\,\mathrm{K})^4 \\
&= (6.315724\ldots \times 10^7\,\mathrm{W/m^2}). \\
\therefore \quad R_{T_\odot} &\approx 6.316 \times 10^7\,\mathrm{W/m^2}.
\end{aligned}
$$

   (b) Given that the Sun is a sphere of radius $R_\odot = 6.95508 \times 10^8\,\mathrm{m}$, what is its total power output? [5]

   Solution: The radiancy multiplied by the surface area or the spherical Sun will give us the Sun's total power output, thus

$$
\begin{aligned}
P_\odot &= R_{T_\odot} A_\odot = \left(\sigma T_\odot^4\right)\left(4\pi R_\odot^2\right) \\
&= (6.315724\ldots \times 10^7\,\mathrm{W/m^2})(6.0787\ldots \times 10^{18}\,\mathrm{m^2}) \\
&= (3.83916\ldots \times 10^{26}\,\mathrm{W}). \\
\therefore \quad P_\odot &\approx 3.839 \times 10^{26}\,\mathrm{W}.
\end{aligned}
$$

   (c) How much solar energy per second passes through an area of one square metre at a distance $r_\oplus = 1.496 \times 10^{11}\,\mathrm{m}$ from the Sun's centre (this is the average Earth-Sun distance)? [5]

   Solution: This power density $\rho_{r_\oplus}$ is calculated by taking the total solar power output $P_\odot$ and dividing it by the surface area of the sphere of radius $r_\oplus$.

$$
\begin{aligned}
\rho_{r_\oplus} &= \frac{P_\odot}{A_{r_\oplus}} = \frac{\left(\sigma T_\odot^4\right)\left(4\pi R_\odot^2\right)}{4\pi r_\oplus^2} = \frac{\sigma T_\odot^4 R_\odot^2}{r_\oplus^2} \\
&= (1365.099\ldots\ \mathrm{W/m^2}). \\
\therefore \quad \rho_{r_\oplus} &\approx 1365\,\mathrm{W/m^2}.
\end{aligned}
$$

   As a good rule of thumb, one square meter of sunlight provides about 1000 W of power. Solar electric panels only take advantage of a small fraction of this power, however pretty much all of it gets converted into heat when it shines into a window. This is why one of the most effective way of keeping a house cool in the summer is to close the curtains and plant shade trees – 1000 W is a lot of power for each square meter of sun exposure.

(d) Suppose that Earth (radius $R_\oplus = 6.37814 \times 10^6$ m) absorbs all incident sunlight (100%), and reradiates as a blackbody. What would be its temperature T_{100}? [5]

Solution: The Earth blocks out an area of $\pi r_\oplus^2$, and so this is the area we will use to find the power that is absorbed by the Earth. Note that we do not use the whole surface area of the spherical Earth, since the whole Earth does not intercept the sunlight all at the same time. We also do not use half of the whole surface area of the spherical Earth, since even though half of the Earth is in daylight, the areas that are not perpendicular to the rays of sunlight should not be given as much importance as the areas that are perpendicular to the rays of sunlight. We need to use the two dimensional area blocked out by the spherical Earth, and since a sphere always casts a circular shadow, we need only find the area of a circle of radius $r_\oplus$.

The power input to the Earth is given by:

$$\begin{aligned}
P_{100\oplus} &= \rho_{r_\oplus} A_{\text{circle}\oplus} \\
&= \frac{\sigma T_\odot^4 R_\odot^2}{r_\oplus^2} \left(\pi R_\oplus^2\right) \\
&= (1365.099\ldots\ \mathrm{W/m^2})(1.2780\ldots \times 10^{14}\ \mathrm{m^2}) \\
P_{100\oplus} &= (1.74462\ldots \times 10^{17}\ \mathrm{W}).
\end{aligned}$$

To find the radiancy of the Earth, we must note that over the long term, the Earth will emit just as much power as it absorbs, otherwise it will heat up or cool down until it is in equilibrium. To calculate the radiancy we need to divide this incoming/outgoing power by the total surface area of the spherical Earth.

$$\begin{aligned}
R_{T100\oplus} &= \frac{P_{100\oplus}}{A_{\text{sphere}\oplus}} = \frac{\sigma T_\odot^4 R_\odot^2}{r_\oplus^2} \frac{\left(\pi R_\oplus^2\right)}{\left(4\pi R_\oplus^2\right)} \\
&= \frac{\sigma T_\odot^4 R_\odot^2}{4 r_\oplus^2} \\
R_{T100\oplus} &= (341.27\ldots\ \mathrm{W/m^2})
\end{aligned}$$

Note that the radiancy is independent of the size of the Earth!

We can again apply Stefan's law to obtain the temperature of the Earth as a blackbody and we get

$$\begin{aligned}
T_{100\oplus}^4 &= \frac{R_{T100\oplus}}{\sigma} = \frac{\sigma T_\odot^4 R_\odot^2}{\sigma 4 r_\oplus^2} \\
&= \frac{T_\odot^4 R_\odot^2}{4 r_\oplus^2} \\
T_{100\oplus}^4 &= (6.018531\ldots \times 10^9\ {}^\circ\mathrm{K}^4) \\
T_{100\oplus} &= \sqrt[4]{T_{100\oplus}^4} = \sqrt[4]{\frac{T_\odot^4 R_\odot^2}{4 r_\oplus^2}} = T_\odot \sqrt{\frac{R_\odot}{2 r_\oplus}} \\
&= 278.530\ldots\ {}^\circ\mathrm{K} \\
\therefore \quad T_{100\oplus} &\approx 278.5\ {}^\circ\mathrm{K}.
\end{aligned}$$

Note that the temperature calculated depends only on the size, distance from, and temperature of the Sun, not on the size of the Earth. This might seem like a very high temperature unless we recall that water freezes at about 273°K, so this is only a few degrees above °C, which does not seem too unreasonable.

(e) Suppose that Earth absorbs 65% of incident sunlight, and reradiates as a blackbody. What would be its temperature T_{65}? Where is the contradiction in this last argument? [5]

Solution: If it only absorbs 65%, then the power input $P_{65\oplus}$ will be 65% of $P_{100\oplus}$, thus the radiancy $R_{T65\oplus}$ will be 65% of $R_{T100\oplus}$ and the fourth power of the temperature $T^4_{65\oplus}$ will be 65% of $T^4_{100\oplus}$. Thus for the final result we have that

$$\begin{aligned}
T^4_{65\oplus} &= \frac{65}{100} T^4_{100\oplus} \\
T_{65\oplus} &= \sqrt[4]{\frac{65}{100}} T_{100\oplus} \\
&= \sqrt[4]{\frac{65}{100}} T_{\odot} \sqrt{\frac{R_{\odot}}{2r_{\oplus}}} \\
&= (0.89790\ldots)\,(278.530\ldots\,{}^\circ\mathrm{K}) \\
&= (250.092\ldots\,{}^\circ\mathrm{K}) \\
\therefore \quad T_{65\oplus} &\approx 250.1\,{}^\circ\mathrm{K}.
\end{aligned}$$

This gives a temperature of about -20°C, which seems a bit low to me. One contradiction in using only 65% absorption is that in order for the black body results to apply, it should be completely absorbing, otherwise it would not be "black". If we were visitors to the local solar system and were interested in trying to measure the temperature of the Earth by examining its black body radiation, we would need to make sure that we took into account the difference between the 35% (=100-65) reflection of solar radiation that this question implies. Perhaps doing the measurements only on the night time side of the planet would address this type of issue, but that would introduce the added complication that the night time temperature is usually different than the day time temperature.

Headstart for next week, Week 03, starting Monday 2004/09/27:
– Read Chapter 2.2 "Light as a Particle" in "Simple Nature" by Crowellk
– Read Chapter 2 "Photons – Particlelike Properties of Radiation" in Eisberg & Resnick
– – Section 2.1 "Introduction"
– – Section 2.2 "The Photoelectric Effect"
– – Section 2.3 "Einstein's Quantum Theory of the Photoelectric Effect"
– – Section 2.4 "The Compton Effect"
– – Section 2.5 "The Dual Nature of Electromagnetic Radiation"
– – Section 2.6 "Photons and X-Ray Production"