1. 9.7-1 Without using your OR Courseware, consider the minimum cost flow problem shown below, where the $b_i$ values (net flows generated) are given by the nodes, the $c_{ij}$ values (costs per unit flow) are given by the arcs, and the $u_{ij}$ values (arc capacities) are given to the right of the network.

(a) Obtain an initial BF solution by solving the feasible spanning tree with basic arcs $A \to B$, $C \to E$, $D \to E$, and $C \to A$ (a reverse arc), where one of the nonbasic arcs ($C \to B$) also is a reverse arc. Show the resulting network (including $b_i$, $c_{ij}$, and $u_{ij}$) in the same format as the above one (except use dashed lines to draw the nonbasic arcs), and add the flows in parentheses next to the basic arcs.

(b) Use the optimality test to verify that this initial BF solution is optimal and that there are multiple optimal solutions. Apply one iteration of the network simplex method to find the other optimal BF solution, and then use these results to identify the other optimal solutions that are not BF solutions.
(c) Now consider the following BF solution.

<table>
<thead>
<tr>
<th>Basic Arc</th>
<th>Flow</th>
<th>Nonbasic Arc</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A \to D$</td>
<td>20</td>
<td>$A \to B$</td>
</tr>
<tr>
<td>$B \to C$</td>
<td>10</td>
<td>$A \to C$</td>
</tr>
<tr>
<td>$C \to E$</td>
<td>10</td>
<td>$B \to D$</td>
</tr>
<tr>
<td>$D \to E$</td>
<td>20</td>
<td></td>
</tr>
</tbody>
</table>

Starting from this BF solution, apply one iteration of the network simplex method. Identify the entering basic arc, the leaving basic arc, and the next BF solution, but do not proceed further.

Solution:

(a)

(b) Nonbasic arc | unique cycle | $\Delta Z$ when $\theta = 1$
---|---|---
$CB$ | $CB \to BA \to AC$ | $3 - 2 - (-6) = 7 > 0$
$AD$ | $AD \to DE \to EC \to CA$ | $5 + 4 - 3 - 6 = 0$
$BD$ | $BD \to DE \to EC \to CA \to AB$ | $5 + 4 - 3 - 6 + 2 = 2 > 0$

Since all $\Delta Z \geq 0$, the current solution is optimal. Because the arc $AD$ had $\Delta Z = 0$, we let $AD$ be the entering BV to find another optimal solution. To send a flow of $\theta$ unit along the cycle $AD \to DE \to EC \to CA$, we need

- $AD \quad \theta < \infty$
- $DE \quad \theta < \infty$
- $EC \quad \theta \leq 30$
- $CA \quad \theta \leq 10 - 5 = 5$
The new diagram is

![Diagram](image)

Arc capacities
- $A \to C: 10$
- $C \to B: 25$
- Others: $\infty$

All the other optimal solutions that are not BF solutions can be obtained by sending a flow of $\theta$ unit along the cycle $AD \to DE \to EC \to CA$ for some $0 < \theta < 5$. That is, $x_{AC} = 5 - \theta$, $x_{AD} = \theta$, $x_{AB} = 15$, $x_{BC} = 25$, $x_{BD} = 0$, $x_{CE} = 30 - \theta$, $x_{DE} = \theta$.

(c)

<table>
<thead>
<tr>
<th>Nonbasic arc</th>
<th>Unique cycle</th>
<th>$\Delta Z$ when $\theta = 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$AB$</td>
<td>$AB \to BC \to CE \to ED \to CB$</td>
<td>$2 + 3 + 3 - 4 - 5 = -1 &lt; 0$</td>
</tr>
<tr>
<td>$AC$</td>
<td>$AC \to CE \to ED \to DA$</td>
<td>$6 + 3 - 4 - 5 = 0$</td>
</tr>
<tr>
<td>$BD$</td>
<td>$BD \to DE \to EC \to CB$</td>
<td>$5 + 4 - 3 - 3 = 3 &gt; 0$</td>
</tr>
</tbody>
</table>

$AB$ is the entering BV. To send a flow of $\theta$ unit along the cycle $AB \to BC \to CE \to ED \to CB$, we need

- $AB: \theta < \infty$
- $BC: \theta \leq 15$
- $CE: \theta \leq \infty$
- $ED: \theta \leq 20$
- $DA: \theta \leq 20$

The minimum is 10. Therefore $BC$ is the leaving BV. (it be-
comes a reversed arc) and the next BF solution is

![Diagram of network with arcs and capacities]

Arc capacities

- $A \rightarrow C$: 10
- $B \rightarrow C$: 25
- Others: $\infty$

2. 9.7-7 Consider the transportation problem having the following cost and requirements table:

<table>
<thead>
<tr>
<th>Source</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>Supply</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6</td>
<td>7</td>
<td>4</td>
<td>40</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>8</td>
<td>6</td>
<td>60</td>
</tr>
<tr>
<td>Demand</td>
<td>30</td>
<td>40</td>
<td>30</td>
<td></td>
</tr>
</tbody>
</table>

Formulate the network representation of this problem as a minimum cost flow problem. Use the northwest corner rule to obtain an initial BF solution. Then use the network simplex method to solve the problem.

Solution: The minimum cost flow problem:
The initial BF solution corresponding to the solution obtained by using the northwest corner rule is

\[
\begin{bmatrix}
30 & 40 \\
60 & & & & 30 \\
& & & & 40 \\
& & & & 30 \\
& & & & 30 \\
\end{bmatrix}
\]

For \( S_1 \to D_3 \), \( \Delta Z = 4 - 6 + 8 - 7 = -1 \). For \( S_2 \to D_1 \), \( \Delta Z = 5 - 6 + 7 - 8 = -2 \). \( S_2 \to D_1 \) is the entering BV, \( S_1 \to D_1 \) is the leaving BV. \( \theta = 30 \). The new BF solution is

\[
\begin{bmatrix}
40 & 6 \\
6 & & & & (30) \\
7 & (10) & & & \\
4 & & & & (40) \\
5 & & & & (30) \\
\end{bmatrix}
\]

For \( S_1 \to D_1 \), \( \Delta Z = 6 - 5 + 8 - 7 = 2 \). For \( S_1 \to D_3 \), \( \Delta Z = 4 - 6 + 8 - 7 = -1 \). \( S_1 \to D_3 \) is the entering BV and \( S_2 \to D_3 \) is the
leaving BV. $\theta = 30$. The new BF solution is

$$\begin{align*}
S &\rightarrow D_1 \quad [30] \\
S &\rightarrow D_2 \quad [40] \\
S &\rightarrow D_3 \quad [60]
\end{align*}$$

This solution is optimal with a total cost of 580.

3. 9.7-8 Consider the minimum cost flow problem shown below, where the $b_i$ values are given by the nodes, the $c_{ij}$ values are given by the arcs, and the finite $u_{ij}$ values are given in parentheses by the arcs. Obtain an initial BF solution by solving the feasible spanning tree with basic arcs $A \rightarrow C$, $B \rightarrow A$, $C \rightarrow D$, and $C \rightarrow E$, where one of the nonbasic arc $(D \rightarrow A)$ is a reverse arc. Then use the network simplex method to solve this problem.
Solution: The initial BF solution is

Nonbasic arc  unique cycle  \( \Delta Z \) when \( \theta = 1 \)

\( DA \)  \( DA \rightarrow AC \rightarrow CD \) \( -6 + 4 + 3 = 1 \)
\( BC \)  \( BC \rightarrow CA \rightarrow AB \) \( 2 - 4 - 1 = -3 \)
\( BE \)  \( BE \rightarrow EC \rightarrow CA \rightarrow AB \) \( 5 - 5 - 4 - 1 = -5 \)

\( BE \) is the entering BV. It is the leaving BV as well (reversed). The next BF solution is

Nonbasic arc  unique cycle  \( \Delta Z \) when \( \theta = 1 \)

\( DA \)  \( DA \rightarrow AC \rightarrow CD \) \( -6 + 4 + 3 = 1 \)
\( BC \)  \( BC \rightarrow CA \rightarrow AB \) \( 2 - 4 - 1 = -3 \)
\( EB \)  \( EB \rightarrow BA \rightarrow AC \rightarrow CE \) \( -5 + 1 + 4 + 5 = 5 \)
BC is the entering BV. BA is the leaving BV. The next BF solution is

Nonbasic arc unique cycle $\Delta Z$ when $\theta = 1$
\begin{align*}
DA & \quad DA \rightarrow AC \rightarrow CD & -6 + 4 + 3 = 1 \\
BA & \quad BA \rightarrow AC \rightarrow CB & 1 + 4 - 2 = 3 \\
EB & \quad EB \rightarrow BC \rightarrow CE & -5 + 2 + 5 = 2
\end{align*}

The current solution is optimal. It corresponds to the real flows with a cost of 750.

4. 10.3-12 Consider the following integer nonlinear programming problem.

Maximize $Z = 18x_1 - x_1^2 + 20x_2 + 10x_3$

subject to

$2x_1 + 4x_2 + 3x_3 \leq 11$
and
\[ x_1, \ x_2, \ x_3 \text{ are nonnegative integers.} \]

Use dynamic programming to solve this problem.

**Solution:** Number of stages = 3. For \( n = 1, 2, 3 \), at stage \( n \), we determine the value of \( x_n \). Let \( s_n \) be the remaining slack in the functional constraint, i.e.,

\[
s_n = \begin{cases} 
11 & n = 1 \\
\ s_1 - 2x_1 & n = 2 \\
\ s_2 - 4x_2 & n = 3 
\end{cases}
\]

We have

\[
f_3^* (s_3) = \max_{x_3=0,1,\ldots,[s_3/3]} 10x_3
\]

<table>
<thead>
<tr>
<th>( s_3 )</th>
<th>( f_3^* (s_3) )</th>
<th>( x_3^* )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>10</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>10</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>10</td>
<td>1</td>
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<tr>
<td>6</td>
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<td>7</td>
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<td>2</td>
</tr>
<tr>
<td>8</td>
<td>20</td>
<td>2</td>
</tr>
<tr>
<td>9</td>
<td>30</td>
<td>3</td>
</tr>
<tr>
<td>10</td>
<td>30</td>
<td>3</td>
</tr>
<tr>
<td>11</td>
<td>30</td>
<td>3</td>
</tr>
</tbody>
</table>

\[
f_2^*(s_2) = \max_{x_2=0,1,\ldots,[s_2/4]} 20x_2 + f_3^*(s_2 - 4x_2)
\]
The optimal solution is $(x_1, x_2, x_3) = (4, 0, 1)$ with $Z = 66$. \(\square\)

5. 10.3-21 Consider the following nonlinear programming problem.

Maximize $Z = x_1 (1 - x_2) x_3,$

subject to

$x_1 - x_2 + x_3 \leq 1$

and

$x_1 \geq 0, \ x_2 \geq 0, \ x_3 \geq 0.$

Use dynamic programming to solve this problem.

**Solution:** (Based on the work of Sarah Jones) In stage $i$ we will determine the value of $x_i$. The state of stage $i$ is $S_i$ where $S_i$ is the slack in the constraint.

\[
\begin{align*}
S_1 &= 1 \\
S_2 &= 1 - x_1 \\
S_3 &= S_2 + x_2
\end{align*}
\]
At stage 3 we
\[
\max f_3(S_3, x_3) = x_3
\]
s.t.
\[
x_3 \leq S_3
\]
The optimal solution is \( x_3^* = S_3 \) and \( f_3^*(S_3) = S_3 \).

At stage 2 we
\[
\max f_2(S_2, x_2) = (1 - x_2) f_3^*(S_2 + x_2)
\]
s.t.
\[
-x_2 \leq S_2 \text{ and } x_2 \geq 0.
\]
Since \( f_2 = S_2 - S_2 x_2 + x_2 - x_2^2 \), \( f_2' = -S_2 + 1 - 2x_2 \). \( f_2' = 0 \implies x_2 = \frac{1 - S_2}{2} \).

Now we consider the constraints. \( S_2 = 1 - x_1 \leq 1 \), therefore \( \frac{1 - S_2}{2} \geq 0 \) is always true. The other constraint is \( -x_2 \leq S_2 \) or \( x_2 \geq -S_2 \).

\[
\frac{1 - S_2}{2} < -S_2 \implies S_2 < -1
\]

So in the case of \( S_2 < -1 \) we will take the left endpoint (which is closer to \( \frac{1 - S_2}{2} \)) \( -S_2 \).

\[
x_2^* = \begin{cases} 
\frac{1 - S_2}{2} & \text{if } S_2 \geq -1; \\
-S_2 & \text{if } S_2 < -1
\end{cases}
\]

and
\[
f_2^*(S_2) = \begin{cases} 
\frac{1}{4} (1 + S_2)^2 & \text{if } S_2 \geq -1; \\
0 & \text{if } S_2 < -1
\end{cases}
\]

At stage 1 we
\[
\max f_1(x_1) = x_1 f_2^*(1 - x_1)
\]
s.t.
\[
x_1 \geq 0
\]

Note here that it is not necessary for \( x_1 \) to be less than or equal to 1 for \( x_2 \) is negative in the original constraint. \( S_1 = 1 \) is not used. Since \( S_2 = 1 - x_1 \), \( S_2 \geq -1 \iff x_1 \leq 2. \)

\[
f_1(x_1) = \begin{cases} 
x_1 \cdot \frac{(2 - x_1)^2}{4} & \text{if } 0 \leq x_1 \leq 2; \\
0 & \text{if } x_1 > 2
\end{cases}
\]
For

\[ f_1(x_1) = x_1 \cdot \frac{(2-x_1)^2}{4} = x_1 - x_1^2 + \frac{1}{4}x_1^3 \]

\[ f_1'(x_1) = 1 - 2x_1 + \frac{3}{4}x_1^2 = \frac{1}{4}(3x_1 - 2)(x_1 - 2) \]

\[ f_1' = 0 \implies x_1 = \frac{2}{3} \text{ or } x_1 = 2 \]

It is easy to check (just check \( x_1 = 0, \frac{2}{3} \text{ and } 2 \)) that \( x_1^* = \frac{2}{3} \) and \( f_1^* = z^* = \frac{8}{27} \). Since \( S_2 = \frac{1}{3} > -1 \), \( x_2^* = \frac{1-S_2}{2} = \frac{1}{3} \) and \( x_3^* = S_3 = S_2 + x_2^* = \frac{2}{3} \).

\[ \square \]

6. 12.2-2 The research and development division of a company has been developing four possible new product lines. Management must now make a decision as to which of these four products actually will be produced and at what levels. Therefore, they have asked the OR department to formulate a mathematical programming model to find the most profitable product mix.

A substantial cost is associated with beginning the production of any product, as given in the first row of the following table. The marginal net revenue from each unit produced is given in the second row of the table.

<table>
<thead>
<tr>
<th>Product</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Start-up cost, $</td>
<td>50,000</td>
<td>40,000</td>
<td>70,000</td>
<td>60,000</td>
</tr>
<tr>
<td>Marginal revenue, $</td>
<td>70</td>
<td>60</td>
<td>90</td>
<td>80</td>
</tr>
</tbody>
</table>

Let the continuous decision variables \( x_1, x_2, x_3 \) and \( x_4 \) be the production levels of products 1, 2, 3, and 4, respectively. Management has imposed the following policy constraints on these variables:

1. No more than two of the products can be produced.
2. Either product 3 or 4 can be produced only if either product 1 or 2 is produced.
3. Either \( 5x_1 + 3x_2 + 6x_3 + 4x_4 \leq 6000 \)
   or \( 4x_1 + 6x_2 + 3x_3 + 5x_4 \leq 6000 \).
Introduce auxiliary binary variables to formulate an MIP model for this problem.

Solution: For \( i = 1, 2, 3, 4 \) let

\[
y_i = \begin{cases} 
1 & \text{if product } i \text{ is produced} \\
0 & \text{otherwise}
\end{cases}
\]

Let \( M \) be a very large number. The problem can be formulated as follows:

\[
\begin{align*}
\text{Maximize } & \quad Z = 70x_1 + 60x_2 + 90x_3 + 80x_4 \\
& \quad -50000y_1 - 40000y_2 - 70000y_3 - 60000y_4
\end{align*}
\]

subject to

\[
\begin{align*}
(1) & \quad \sum_{i=1}^{4} y_i \leq 2 \\
& \quad x_i \leq My_i \quad i = 1, 2, 3, 4 \\
(2) & \quad y_3 + y_4 \leq y_1 + y_2 \\
(3) & \quad 5x_1 + 3x_2 + 6x_3 + 4x_4 \leq 6000 + wM \\
& \quad 4x_1 + 6x_2 + 3x_3 + 5x_4 \leq 6000 + (1 - w)M \\
& \quad x_i \geq 0 \quad i = 1, 2, 3, 4. \ y_i \text{ is binary for } i = 1, 2, 3, 4. \ w \text{ is binary.}
\end{align*}
\]

\[\square\]

7. 12.2-4 Consider the following mathematical model.

\[
\begin{align*}
\text{Maximize } & \quad Z = 3x_1 + 2f(x_2) + 2x_3 + 3g(x_4),
\end{align*}
\]

subject to the restrictions

1. \( 2x_1 - x_2 + x_3 + 3x_4 \leq 15 \).

2. At least one of the following two inequalities holds:

\[
\begin{align*}
x_1 + x_2 + x_3 + x_4 & \leq 4 \\
3x_1 - x_2 - x_3 + x_4 & \leq 3.
\end{align*}
\]
3. At least two of the following four inequalities holds:

\[
\begin{align*}
5x_1 + 3x_2 + 3x_3 - x_4 & \leq 10 \\
2x_1 + 5x_2 - x_3 + 3x_4 & \leq 10 \\
-x_1 + 3x_2 + 5x_3 + 3x_4 & \leq 10 \\
3x_1 - x_2 + 3x_3 + 5x_4 & \leq 10.
\end{align*}
\]

4. \( x_3 = 1, \text{ or } 2, \text{ or } 3. \)
5. \( x_j \geq 0 \) \((j = 1, 2, 3, 4), \)

where

\[
f(x_2) = \begin{cases} 
-5 + 3x_2 & \text{if } x_2 > 0, \\
0 & \text{if } x_2 = 0,
\end{cases}
\]

and

\[
g(x_4) = \begin{cases} 
-3 + 5x_4 & \text{if } x_4 > 0, \\
0 & \text{if } x_4 = 0.
\end{cases}
\]

Formulate this problem as an MIP problem.

**Solution:**

Maximize \( 3x_1 - 10y_1 + 6x_2 + 2x_3 - 9y_2 + 15x_4 \)

subject to

\[
\begin{align*}
x_1 & \leq My_1 \\
x_4 & \leq My_2 \\
2x_1 - x_2 + x_3 + 3x_4 & \leq 15 \\
x_1 + x_2 + x_3 + x_4 & \leq 4 + My_3 \\
3x_1 - x_2 - x_3 + x_4 & \leq 3 + M(1 - y_3)
\end{align*}
\]

\[
\begin{align*}
5x_1 + 3x_2 + 3x_3 - x_4 & \leq 10 + My_4 \\
2x_1 + 5x_2 - x_3 + 3x_4 & \leq 10 + My_5 \\
-x_1 + 3x_2 + 5x_3 + 3x_4 & \leq 10 + My_6 \\
3x_1 - x_2 + 3x_3 + 5x_4 & \leq 10 + My_7 \\
y_4 + y_5 + y_6 + y_7 & = 2
\end{align*}
\]
(4) \[ x_3 = y_8 + 2y_9 + 3y_{10} \]
\[ y_8 + y_9 + y_{10} = 1 \]

(5) \[ x_i \geq 0 \quad i = 1, 2, 3, 4 \text{ and } y_j \text{ binary } j = 1, 2, ..., 10. \]