1. §1.6 #4. (2 points each part) Find the domain and range of the following functions.

(a) the function that assigns to each nonnegative integer its last digit
   \textit{Solution:} Domain: \( \mathbb{N} \); Range: \{0, 1, 2, ..., 9\}.

(b) the function that assigns the next largest integer to a positive integer
   \textit{Solution:} Domain: \( \mathbb{Z}^+ \); Range: \( \mathbb{Z}^+ - \{1\} \).

(c) the function that assigns to a bit string the number of one bits in the string
   \textit{Solution:} Domain: \{all bit strings\}; Range: \( \mathbb{N} \).

(d) the function that assigns to a bit string the number of bits in the string
   \textit{Solution:} Domain: \{all bit strings\}; Range: \( \mathbb{N} \).

\[ \square \]

2. #10. (2 points each part) Determine whether each of the following functions from \( \mathbb{Z} \) to \( \mathbb{Z} \) is one-to-one.

(a) \( f(n) = n - 1 \).
   \textit{Solution:} Yes, since \( n - 1 = n' - 1 \rightarrow n = n' \).

(b) \( f(n) = n^2 + 1 \).
   \textit{Solution:} No, since \( f(-1) = f(1) \).

(c) \( f(n) = n^3 \).
   \textit{Solution:} Yes, since \( n^3 = (n')^3 \rightarrow n = n' \).

(d) \( f(n) = \lceil n/2 \rceil \).
   \textit{Solution:} No, since \( f(1) = f(2) = 1 \).

3. #12. (2 points each part) Give an example of a function from \( \mathbb{N} \) to \( \mathbb{N} \) that is

(a) one-to-one but not onto.
   \textit{Solution:} \( f(n) = n + 1 \). It is one-to-one since \( n + 1 = n' + 1 \rightarrow n = n' \). It is not onto since 0 is not in the range.

(b) onto but not one-to-one.
   \textit{Solution:} \( f(n) = \lfloor n/2 \rfloor \). It is onto since \( f(2n) = n \) for every \( n \in \mathbb{N} \). It is not one-to-one since \( f(2) = f(3) = 1 \).

(c) both onto and one-to-one (but different from the identity function).
   \textit{Solution:} \( f(n) = \begin{cases} n + 1 & \text{if } n \text{ is even}; \\ n - 1 & \text{if } n \text{ is odd}. \end{cases} \)
   It is onto since if \( n \) is even, \( n+1 \) is odd and \( f(n+1) = n \); if \( n \) is odd then \( n-1 \) is even and \( f(n-1) = n \). It is one-to-one since \( f(n) = f(n') \rightarrow n = n' \).
   It is not the identity function since \( f(1) = 0 \).

(d) neither one-to-one nor onto.
   \textit{Solution:} \( f(n) = 0 \). It is obviously not one-to-one or onto.

4. #24. (2 points) Let \( f(x) = ax + b \) and \( g(x) = cx + d \) where \( a, b, c \) and \( d \) are constants. Determine for which constants \( a, b, c \) and \( d \) it is true that \( f \circ g = g \circ f \).
   \textit{Solution:} \( f \circ g = f(g(x)) = a(cx + d) + b = acx + (ad + b) \) and \( g \circ f = g(f(x)) = c(ax + b) + d = cac + (cb + d) \). If \( f \circ g = g \circ f \), then we would have \( ad + b = cb + d \)

or \( (a - 1)d = (c - 1)b \).
5. #30. (4 points each part) Let $f$ be a function from $A$ to $B$. Let $S$ and $T$ be subsets of $B$. Show that

(a) $f^{-1}(S \cup T) = f^{-1}(S) \cup f^{-1}(T)$

Proof: Let $f(x) = y$. First we show that

$$f^{-1}(S \cup T) \subseteq f^{-1}(S) \cup f^{-1}(T).$$  \hspace{1cm} (1)

Suppose that $x \in f^{-1}(S \cup T)$. $y \in S \cup T$. $y \in S$ or $y \in T$. If $y \in S$, $x \in f^{-1}(S)$ so $x \in f^{-1}(S) \cup f^{-1}(T)$. If $y \in T$, $x \in f^{-1}(T)$ so $x \in f^{-1}(S) \cup f^{-1}(T)$. In either case (1) is true.

Then we show that

$$f^{-1}(S \cup T) \supseteq f^{-1}(S) \cup f^{-1}(T).$$  \hspace{1cm} (2)

Suppose that $x \in f^{-1}(S) \cup f^{-1}(T)$. Then $x \in f^{-1}(S)$ or $x \in f^{-1}(T)$. If $x \in f^{-1}(S)$ then $y \in S$ so $y \in S \cup T$. Therefore $x \in f^{-1}(S) \cup f^{-1}(T)$. If $x \in f^{-1}(T)$ then $y \in T$ so $y \in S \cup T$. Therefore $x \in f^{-1}(S) \cup f^{-1}(T)$. In either case (2) is true. Combining (1) and (2), we have $f^{-1}(S \cup T) = f^{-1}(S) \cup f^{-1}(T)$.

(b) $f^{-1}(S \cap T) = f^{-1}(S) \cap f^{-1}(T)$

Proof: This is similar to the proof in part (a). Let $y = f(x)$. First we show that

$$f^{-1}(S \cup T) \subseteq f^{-1}(S) \cup f^{-1}(T).$$  \hspace{1cm} (3)

Suppose that $x \in f^{-1}(S \cup T)$ then $y \in S \cup T$. Either $y \in S$ or $y \in T$. If $y \in S$ then $x \in f^{-1}(S)$ and $x \in f^{-1}(S) \cup f^{-1}(T)$. If $y \in T$ then $x \in f^{-1}(T)$ and $x \in f^{-1}(S) \cup f^{-1}(T)$. Therefore (3) is true. Then we show that

$$f^{-1}(S) \cup f^{-1}(T) \subseteq f^{-1}(S \cup T).$$  \hspace{1cm} (4)

Suppose that $x \in f^{-1}(S) \cup f^{-1}(T)$ then $x \in f^{-1}(S)$ or $x \in f^{-1}(T)$. If $x \in f^{-1}(S)$ then $y \in S \rightarrow y \in S \cup T \rightarrow x \in f^{-1}(S \cup T)$. If $x \in f^{-1}(T)$ then $y \in T \rightarrow y \in S \cup T \rightarrow x \in f^{-1}(S \cup T)$. Therefore (4) is true. Combining (3) and (4), we have $f^{-1}(S \cap T) = f^{-1}(S) \cap f^{-1}(T)$. \hfill \box

6. §1.7 #10. (3 points each part) For each of the following lists of integers, provide a simple formula or rule that generates the terms of an integer sequence that begins with the given list.

(a) 3, 6, 11, 18, 27, 38, 51, 66, 83, 102, ...

Solution: $f(n) = n^2 + 2$, $n = 1, 2, 3, ...$

(b) 7, 11, 15, 19, 23, 27, 31, 35, 39, 43, ...

Solution: $f(n) = 4n + 3$, $n = 1, 2, 3, ...$

(c) 1, 10, 11, 100, 101, 110, 111, 1000, 1001, 1010, 1011, ....

Solution: $f(n) = n$ in binary.

(d) 1, 2, 2, 2, 3, 3, 3, 3, 5, 5, 5, 5, 5, 5, ...

Solution: The nth prime repeated $2n - 1$ times. (Strictly speaking, this is not right. Since 1 is not a prime. Precisely, this should be “the first entry is 1 and then the nth prime repeated $2n + 1$ times”)

(e) 0, 2, 8, 26, 80, 242, 728, 2186, 6560, 19682, ...

Solution: $f(n) = 3^n - 1$.

(f) 1, 3, 15, 105, 945, 10395, 135135, 2027025, 34459425, ...

Solution: $1 \times 3, 1 \times 3 \times 5, 1 \times 3 \times 5 \times 7, \ldots \ f(n) = 1 \times 3 \times \cdots \times (2n - 1)$.

(g) 1, 0, 1, 1, 0, 0, 0, 1, 1, 1, 1, 1, ...

Solution: One 1, followed by 2 zeroes, followed by 3 1s, followed by 4 zeroes, ...

(h) 2, 4, 16, 256, 65536, 4294967296, ...

Solution: $f(n) = 2^{f(n-1)} = 2^{\sum_{k=1}^{n-1} 2^k}$ of them
7. #14. (1 point each part) What are the values of the following sums, where \( S = \{1, 3, 5, 7\} \)?

(a) \( \sum_{j \in S} j \)
Solution: \( \sum_{j \in S} j = 1 + 3 + 5 + 7 = 16 \)

(b) \( \sum_{j \in S} j^2 \)
Solution: \( \sum_{j \in S} j^2 = 1^2 + 3^2 + 5^2 + 7^2 = 84 \)

(c) \( \sum_{j \in S} \frac{1}{j} \)
Solution: \( \sum_{j \in S} \frac{1}{j} = \frac{1}{1} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} = \frac{176}{105} \)

(d) \( \sum_{j \in S} \frac{1}{j^2} \)
Solution: \( \sum_{j \in S} \frac{1}{j^2} = 1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} = \frac{955}{343} \).

8. (4 points) Supplementary Exercises of Chapter 1 #32. Show that if \( n \) is an integer, then \( n = \lfloor n/2 \rfloor + \lceil n/2 \rceil \).
Proof: Case 1. \( n \) is even, i.e., \( n = 2k \) for some integer \( k \). In this case, \( \lfloor n/2 \rfloor + \lceil n/2 \rceil = \lfloor 2k/2 \rfloor + \lceil 2k/2 \rceil = k + k = 2k = n \).
Case 2. \( n \) is odd, i.e., \( n = 2k + 1 \) for some integer \( k \). In this case, \( \lfloor n/2 \rfloor + \lceil n/2 \rceil = \lfloor (2k + 1)/2 \rfloor + \lceil (2k + 1)/2 \rceil = k + 1 + k = 2k + 1 = n \).

9. (4 points) Supplementary Exercises of Chapter 1 #34. Is the set of irrational numbers between 0 and 1 countable? Justify your answer.
Solution: The set of irrational numbers between 0 and 1 is not countable. Suppose that it is. We know that all the rational numbers between 0 and 1 are countable. Then we can have a list of the rational numbers between 0 and 1: \( r_0, r_1, ..., r_k, ... \) and a list of the irrational numbers between 0 and 1: \( i_0, i_1, ..., i_k, ... \). And \( r_0, i_0, r_1, i_1, ..., r_k, i_k, ... \) would be a list of all the real numbers between 0 and 1. However, we know that all the real numbers between 0 and 1 form a set that is not countable. This is a contradiction.

10. §3.1 #18. (2 points each part) Prove that if \( n \) is an integer and \( 3n + 2 \) is even, then \( n \) is even using
(a) an indirect proof.
Proof: Suppose that \( n \) is not even. Then \( n \) is odd, i.e., \( n = 2k + 1 \) for some integer \( k \). Then
\[
3n + 2 = 3(2k + 1) + 2 = 6k + 3 + 2 = 6k + 4 + 1 = 2(3k + 2) + 1
\]
is not even.

(b) a proof by contradiction.
Proof: Assume that \( 3n + 2 \) is even and \( n \) is odd. We can show that \( 3n + 2 \) is odd as in part (a). Thus \( 3n + 2 \) is even and odd, that is a contradiction.

11. #22. (3 points) Prove that the product of two rational numbers is rational.
Proof: Suppose that \( x = \frac{a}{b} \) and \( y = \frac{c}{d} \) are two rational numbers where \( a, b, c, d \) are integers. Then the product
\[
xy = \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}
\]
is rational since both \( ac \) and \( bd \) are integers.

12. #24. (2 points) Prove or disprove that the product of a nonzero rational number and an irrational number is irrational.
Proof: We prove the product is irrational using indirect proof. Suppose that \( x = \frac{p}{q} \) for some integers \( a \) and \( b \) where \( a \neq 0 \). \( y \) is an irrational number and \( z = xy \). Suppose that \( z \) is a rational number. Then \( z = \frac{c}{d} \) for some integers \( c \) and \( d \). We have

\[
\frac{c}{d} = \frac{a}{b} \cdot y \\
\rightarrow y = \frac{cd}{ad}
\]

Since both \( cd \) and \( ad \) are integers, \( y \) is rational.

13. #38. (2 points) Prove or disprove that \( n^2 - 1 \) is composite whenever \( n \) is a positive integer greater than 1.

Proof: The statement is not true since it is not true for \( n = 2: 2^2 - 1 = 3 \) is a prime.

14. #62. (2 points) Prove or disprove that if \( a \) and \( b \) are rational numbers, then \( ab \) is also rational.

Proof: The statement is not true since \( a = 2, b = \frac{1}{2} \) is a counter example: \( 2 \frac{1}{2} = \sqrt{2} \) is irrational.

15. §3.2 #8. (4 points) Show that \( 1^3 + 2^3 + \cdots + n^3 = [n (n + 1)]^2 / 4 \) whenever \( n \) is a positive integer.

Proof: Base step: \( P(1): 1^3 = [1 (1 + 1)]^2 / 2 = 1 \) is true.

Inductive step: Assume that \( P(n) \) is true: \( 1^3 + 2^3 + \cdots + n^3 = [n (n + 1)]^2 / 4 \). We show that \( P(n + 1): 1^3 + 2^3 + \cdots + n^3 + (n + 1)^3 = [(n + 1) (n + 2)]^2 / 4 \) is true.

The

\[
\text{LHS} = 1^3 + 2^3 + \cdots + n^3 + (n + 1)^3 \\
= \frac{n^2 (n + 1)^2}{4} + (n + 1)^3 = \frac{(n + 1)^2 [n^2 + 4 (n + 1)]}{4} \\
= \frac{(n + 1)^2 (n + 2)^2}{4} = \text{RHS}
\]

16. #14. (4 points) Use mathematical induction to prove that \( n! < n^n \) whenever \( n \) is a positive integer greater than 1.

Proof: Base step: \( P(2): 2! < 2^2 \) is true since \( 2! = 2 \) and \( 2^2 = 4 \).

Inductive step: Assume that \( P(n) \) is true: \( n! < n^n \). We show that \( P(n + 1): (n + 1)! < (n + 1)^{n+1} \) is true.

\[
\text{LHS} = (n + 1)! = (n + 1) \cdot n! < (n + 1) \cdot n^n \\
< (n + 1) \cdot (n + 1)^{n+1} = (n + 1)^{n+1} = \text{RHS}.
\]

17. #20. (4 points) Use mathematical induction to show that \( 3 \) divides \( n^3 + 2n \) whenever \( n \) is a nonnegative integer.

Proof: Base step: \( P(0): 3 \) divides \( 0^3 + 2 \cdot 0 = 0 \) is true.

Inductive step: Assume that \( P(n): 3 \) divides \( n^3 + 2n \) is true. We show that \( P(n + 1) \) is true, i.e., \( 3 \) divides \( (n + 1)^3 + 2(n + 1) \).

Since \( 3 \) divides \( n^3 + 2n \), we have \( n^3 + 2n = 3k \) for some integer \( k \). Then

\[
(n + 1)^3 + 2(n + 1) = n^3 + 3n^2 + 3n + 1 + 2n + 2 \\
= n^3 + 3n^2 + 3n + 1 + 2n + 2 \\
= 3k + 3n^2 + 3n + 3 = 3(k + n^2 + n + 1)
\]

Therefore \( 3 \) divides \( (n + 1)^3 + 2(n + 1) \).
18. #28. (4 points) For which nonnegative integers $n$ is $n^2 \leq n!$? Prove your answer using mathematical induction.

Solution: $n^2 \leq n!$ is true for $n = 0, 1$, not true for $n = 2, 3$ and true for $n \geq 4$. We show $n^2 \leq n!$ for all integers $n \geq 4$ using induction.

Base step: $P(4)$ is true since $4^2 = 16 \leq 4! = 24$.

Inductive step: Assume that $P(n)$ is true: $n^2 \leq n!$. We show that $P(n+1)$ is true: $(n+1)^2 \leq (n+1)!$

\[
(n+1)^2 = n^2 + 2n + 1 \leq n! + 2n + 1 \leq n! + n! + n! \leq 3 \cdot n! \leq (n+1) n! = (n+1)!
\]

\[\square\]

19. #30. (4 points) Use mathematical induction to show that

\[\frac{1}{2n} \leq \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdots 2n}\]

Proof: $P(1)$ is true:

\[\frac{1}{2} \leq \frac{1}{2}\]

Assume that $P(n)$ is true:

\[\frac{1}{2n} \leq \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdots 2n}\]

We show that $P(n+1)$ is true:

\[\frac{1}{2(n+1)} \leq \frac{1 \cdot 3 \cdot 5 \cdots (2n-1) (2n+1)}{2 \cdot 4 \cdots 2n \cdot 2(n+1)}\]

\[\begin{align*}
RHS & = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdots 2n} \cdot \frac{2n+1}{2(n+1)} \\
& > \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdots 2n} \cdot \frac{2n}{2(n+1)} \\
& = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdots 2n} \cdot \frac{n}{(n+1)} \\
& \geq \frac{1}{2n} \cdot \frac{n}{(n+1)} = \frac{1}{2(n+1)} = LHS
\end{align*}\]