**Physics 380H - Wave Theory** **Midterm Test - Solutions**

Fall 2004 **Name:** ____________________

Friday 2004/10/22 **Student Number:** ____________________

This examination paper includes 2 pages and 6 questions. You are responsible for ensuring that your copy of the paper is complete. Bring any discrepancy to the attention of your invigilator.

Special Instructions:

The only aids allowed are: a one (1) page single-sided hand-written formula sheet, and a calculator. When completed, turn in all exam booklets, the test paper, and the formula sheet.

Write your name and student number on the top of this paper AND on the front of your answer booklet AND on your formula sheet. Be prepared to present your student ID for verification.

Portable communications devices of all types (e.g. pagers, cellular phones, communicating calculators) are prohibited in the examination room. All such devices must be turned off prior to the start of the examination. A penalty of 5% of the exam mark may be assessed to anyone who fails to prevent a call from interrupting the examination.

Giving or receiving aid during an exam is a violation of university rules and may result in a failing grade and/or expulsion from the university.

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Each part of question one is worth three (3) points, and each of the other questions are worth ten (10) points each. Do all of question one and choose four (4) of the remaining problems to do.

1. In 100 words or less or using algebraic formulae where all of the symbols are defined, or annotated diagrams where appropriate, explain or define each of the following terms of wave phenomena: [21]

   (a) isomorphism

   **Solution:** The word "isomorphism" applies when two complex structures can be mapped onto each other, in such a way that to each part of one structure there is a corresponding part in the other structure, where "corresponding" means that the two parts play similar roles in their respective structures. - Douglas Hofstadter, (Gdel, Escher, Bach, p. 49)

   If there exists an isomorphism between two structures, we call the two structures isomorphic. Isomorphic structures are "the same" at a certain level of abstraction; ignoring the specific identities of the elements in the underlying sets and the names of the underlying relations, the two structures are identical. - http://www.wordiq.com/definition/Isomorphism/

   In the case of waves, we found various systems (strings, acoustic, tidal, etc.) which had mathematical structures which were identical, so we could apply the results from one area directly to another by just translating symbols.

   (b) $p$, $s$, $\xi$, and $\dot{\xi}$ for acoustic waves

   **Solution:** $p(x,t)$ is the unction representing *excess acoustic pressure*, or the pressure relative to the normal undisturbed pressure and is given by $p = P - P_0$.

   $s(x,t)$ is the unction representing *relative* change in density, measured from the undisturbed density, referred to as the *condensation variable*, given by $s = (\rho - \rho_0)/\rho_0$, a measure of how much the fluid is compressed (condensations) or uncompressed (rarefactions).

   $\xi(x,t)$ is the function representing the *displacement* of the particles from their undisturbed position $x$ - this displacement is parallel to the wave motion for acoustic waves.

   $\dot{\xi}(x,t) = \partial\xi(x,t)/\partial t$ is the function representing the *velocity* of the particles originally at the undisturbed position $x$ - this velocity is parallel to the wave motion for acoustic waves.

(c) convective acoustic intensity

**Solution:** An intensity is a measure of power (work/energy per unit time) per unit area. The convective acoustic intensity $I' = P_0\dot{\xi}$ is the rate at which work is done per unit area on a fluid by moving the fluid in bulk in one direction or another, without any disturbance from the normal undisturbed pressure. $I'$ can take positive or negative values depending only on the value of $\dot{\xi}$. For non-dissipative media, for periodic waves (or any situation where $\xi$ returns to the same value), net energy delivered to the system (which is the time integral of $I'$) is zero, and thus the average value of $I'$ is also zero. This is to be contrasted with the *radiative acoustic intensity* $i = p\dot{\xi}$ which can have net energies delivered to the system and average values that are non-zero even for periodic waves.

(d) "3 db down"

**Solution:** A decrease of 3 decibels corresponds with a halving of the intensity. Since the decibel scale is logarithmic (multiplied by ten), decreases (or increases) by factors of two in intensity correspond with decreases (or increases) of $\log_{10} 2$ (multiplied by ten). Since $\log_{10} 2 = 0.3010$, a decrease of 3 db means that the measured intensity is one half of the original (actually it is just a bit less than one half). This analysis applies to any situation where the decibel scale is used - which is most commonly used for sound intensities.

(e) Poynting vector

**Solution:** The Poynting vector $\mathbf{S}$ is the *energy flux* or *intensity* of an electromagnetic wave, or energy per unit time per unit area, and is calculated by the cross product $\mathbf{S} = \mathbf{E} \times \mathbf{H}$, where $\mathbf{E}$ is the electric field vector and $\mathbf{H}$ is the magnetic field vector. $\mathbf{S}$ has a magnitude equal to the power per unit area crossing a surface whose normal is parallel to $\mathbf{S}$ and a direction in the direction of the motion of the electromagnetic wave.

(f) reflection coefficient

**Solution:** The *reflection coefficient* is the factor $R$ by which the amplitude of an incoming wave is reduced upon reflection. Given an incoming wave to a surface, there are two outgoing waves from the surface - the reflected wave and the transmitted wave. By applying the appropriate boundary conditions on $\xi$ and $\dot{\xi}$ in the media on each side of the surface, $R_{\dot{\xi}}$, $R_{\xi}$, $R_p$ and other reflection coefficients can be calculated, in terms of the physical characteristics (ie. the impedances or relative impedances) of the media in question.

(g) sound decibel level

**Solution:** The decibel scale is a logarithmic function of the relative intensity of a sound, measured relative to the quietest sound the average human ear can detect (about $10^{-12}\,\mathrm{w/m^2}$). The sound decibel level is given by $\Delta = 10 \log_{10}\left(\frac{i}{i_0}\right)$. As the threshold for pain is about $1\,\mathrm{w/m^2}$, typical sound levels range from just above 0 decibels to around 120 or 130 decibels. humans are generally able to discriminate a difference in sound decibel levels of slightly more than 1 db.

2. A string which is initially straight is struck with a broad mallet of width $2a$. At time $t = 0$, the mallet imparts an initial velocity $v_0$ to the particles which lie between $x = -a$ and $x = +a$.

$$\dot{y}(x,0) = \psi(x) = \begin{cases} 0, & x < -a, \\ v_0, & -a \le x \le a, \\ 0, & a < x. \end{cases}$$

(a) What is the displacement $y(x,t)$ at times $t > 0$? [5]

**Solution:** This problem is very similar to Towne Section 1.8 Example 2, pg 15. The unique solution to the wave equation subject to the given initial conditions is

$$\begin{aligned} y(x,t) &= f(x-ct) + g(x+ct) \\ &= \frac{1}{2}[\phi(x+ct) + \phi(x-ct) + \chi(x+ct) - \chi(x-ct)], \end{aligned}$$

so the initial condition is that $y(x,0) = \phi(x) = 0$ and $\dot{y}(x,0) = c\chi(x) = \psi(x)$ as above. Integrating the expression for $\psi(x)$ from $-\infty$ to $x$, we find

$$\chi(x) = \frac{1}{c}\int_{-\infty}^{x} \psi(x)\mathrm{d}x = \begin{cases} 0, & x < -a, \\ v_0 x/c, & -a \le x \le a, \\ v_0 2a/c, & a < x. \end{cases}$$

Since $f(x) = \frac{1}{2}[\phi(x) - \chi(x)] = -\frac{1}{2}\chi(x)$, the + wave shape is given by:

$$f(x) = \begin{cases} 0, & x < -a, \\ -v_0 x/2c, & -a \le x \le a, \\ -v_0 a/c, & a < x. \end{cases}$$

Since $g(x) = \frac{1}{2}[\phi(x) + \chi(x)] = \frac{1}{2}\chi(x) = -f(x)$, the - wave shape is given by:

$$g(x) = \begin{cases} 0, & x < -a, \\ v_0 x/2c, & -a \le x \le a, \\ v_0 a/c, & a < x. \end{cases}$$

The overall solution is given by replacing $x$ with $x \pm ct$ in $f$ and $g$ to give:

$$y(x,t) = f(x-ct) + g(x+ct)$$

$$f(x-ct) = \begin{cases} 0, & (x-ct) < -a, \\ -v_0(x-ct)/2c, & -a \le (x-ct) \le a, \\ -v_0 a/c, & (x-ct) < x. \end{cases}$$

$$g(x+ct) = \begin{cases} 0, & (x+ct) < -a, \\ v_0(x+ct)/2c, & -a \le (x+ct) \le a, \\ v_0 a/c, & a < (x+ct). \end{cases}$$

(b) Describe the shapes of the + and - wave components. [3]

**Solution:** As shown above by $f(x)$ and $g(x)$ above, the waves are wedges with slopes of $\pm v_0/2c$, heights of $v_0 a/c$, and having slopes of length of $2a$. The $f$ wedge travelling in the + direction is downward ($-y$ direction) and the $g$ wedge travelling in the - direction is upward ($+y$ direction).

(c) What does the string look like at about time $t \approx a/4c$? (Sketch the graph of $y(x,t)$ at this time.) [2]

**Solution:** At this time the each wave will have moved about one quarter of the distance $a$, so there will be an upward slope (up from zero) from about $-a/4$ to $a/4$, a flat portion of height $v_0 a/8c$ from $a/4$ to $a3/4$ and a slope back down to zero from $a3/4$ to $a5/4$.

3. The velocity distribution in an acoustic place wave is given by the function

$$\dot{\xi}(x,t) = \dot{\xi}_m \sin(kx)\cos(\omega t).$$

(a) What is the corresponding pressure distribution? [6]

**Solution:** The pressure distribution is given by

$$p(x,t) = -\mathcal{B}_a \frac{\partial \xi}{\partial x}(x,t)$$

so we need to intigrate $\dot{\xi}(x,t)$ with respect to $t$, and then differentiate with respect to $x$ in order to get an expression for $p(x,t)$.

$$\begin{aligned}
\xi(x,t) &= \int \dot{\xi}(x,t)\mathrm{d}t = \int \dot{\xi}_m \sin(kx)\cos(\omega t)\mathrm{d}t \\
&= \dot{\xi}_m \sin(kx) \int \cos(\omega t)\mathrm{d}t \\
&= \dot{\xi}_m \sin(kx) \frac{\sin(\omega t)}{\omega} = \frac{\dot{\xi}_m}{\omega} \sin(kx)\sin(\omega t) \\
p(x,t) &= -\mathcal{B}_a \frac{\partial \xi}{\partial x}(x,t) = -\mathcal{B}_a \frac{\partial}{\partial x}\left[\frac{\dot{\xi}_m}{\omega}\sin(kx)\sin(\omega t)\right] \\
&= -\frac{\mathcal{B}_a \dot{\xi}_m}{\omega}\sin(\omega t)\frac{\partial}{\partial x}[\sin(kx)] = -\frac{\mathcal{B}_a \dot{\xi}_m k}{\omega}\sin(\omega t)\cos(kx) \\
&= -\dot{\xi}_m \frac{\mathcal{B}_a k}{\omega}\sin(\omega t)\cos(kx) \\
&= -\dot{\xi}_m \frac{\mathcal{B}_a}{c}\sin(\omega t)\cos(kx) \\
&= -\dot{\xi}_m Z \sin(\omega t)\cos(kx)
\end{aligned}$$

(b) At what values of $x$ could a rigid surface be placed without disturbing the given distributions? [4]

**Solution:** A rigid surface will force the displacements of particles at that position to be zero, thus we could place a rigid surface at any of the nodes of $\xi$, where $\xi(x,t) = 0$, so

$$\xi(x,t) = \frac{\dot{\xi}_m}{\omega}\sin(kx)\sin(\omega t) = 0 \quad \Longrightarrow \quad \sin(kx) = 0$$

$$\Longrightarrow \quad kx = n\pi \quad \Longrightarrow \quad x = n\frac{\pi}{k} = n\frac{c\pi}{\omega} \qquad n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

Thus we have that for any integer $n$, a rigid surface could be placed at $x = n\pi/k$ and the given distributions would remain unchanged. This is the same result that we would have obtained by requiring that the surface was placed at the nodes of $\dot{\xi}(x,t)$.

Alternatively and equivalently we could have performed the calculation by requiring that the rigid surface be placed at the anti-nodes of $p(x,t)$ since rigid surfaces constrain the pressure to be $\pm$maxima in the same way that they constrain the displacement (and displacement velocity) to be zero.

4. Waves on a liquid surface have an isomorphic relationship to acoustic waves, and are called "Tidal Waves" by Towne. Such waves are described by the variables: $h$ (the undisturbed depth of the liquid), $\rho$ (the density of the liquid), $\xi(x,t)$ (the horizontal displacement of particles originally at $x$), $\eta(x,t)$ (the elevation of the surface above the undisturbed depth $h$, corresponding to the plane of particles originally at $x$), and $g$ (the acceleration of gravity).

   The (acoustic $\leftrightarrow$ tidal) isomorphism is: $(\xi \leftrightarrow \xi)$, $(p \leftrightarrow \rho g \eta)$, $(\mathcal{B}_a \leftrightarrow \rho g h)$, and $(\rho_0 \leftrightarrow \rho)$.

   (a) What is $c_{tidal}$, the wave propagation velocity for tidal waves, in terms of the tidal variables? [3]

   **Solution:** $c = \sqrt{\mathcal{B}_a/\rho_0} \leftrightarrow \sqrt{\rho g h/\rho} = \sqrt{gh} = c$

   (b) How many times faster are tidal waves on pure water ($\rho_{fresh} = 998\,\mathrm{kg/m^3}$) than on seawater ($\rho_{salt} = 1025\,\mathrm{kg/m^3}$), for depths of $h = 10\,\mathrm{m}$? [3]

   **Solution:** The speed of tidal waves only depends on the height of the wave and the acceleration of gravity, not on the physical characteristics of the fluid, so the tidal waves on pure water travel at the same speed as the tidal waves of seawater at the given depths, $c_{fresh} = c_{salt}$.

   (c) For a progressive sinusoidal tidal wave moving in the + direction (to the right), the water particles on the surface move in flat elliptical paths. In what direction are the particles moving at the crests and troughs? Equivalently, are they clockwise or counter-clockwise ellipses? [4]

   **Solution:** The motion of the particles is a combination of their vertical velocity given by $\dot{\eta}$ and their horizontal velocity $\dot{\xi}$. At the tops and bottoms of the crests, their vertical speed is zero since they are at the extremes of their motion (they have stopped and are turning around). We want to find out if $\dot{\xi}$ is positive (to the right) or negative (to the left). The isomorphism gives us $(\xi \leftrightarrow \xi)$ and $(p \leftrightarrow \rho g \eta)$, so in the acoustic case we would compare $\dot{\xi}_{acoustic}$ with $\dot{p}_{acoustic}$ and apply that result to the relationship between $\dot{\xi}_{tidal}$ and $\dot{\eta}_{tidal}$.

   For the acoustic case, we know that $p_+ = Z\dot{\xi}_+$, so $\dot{p}_+ = Z\ddot{\xi}_+$, for a sinusoidal wave, such as $\xi = \xi_m \sin(kx - \omega t)$ this gives us:

$$
\begin{aligned}
\xi_+ &= \xi_m \sin(kx - \omega t) \\
\dot{\xi}_+ &= \omega \xi_m \cos(kx - \omega t) \\
\ddot{\xi}_+ &= -\omega^2 \xi_m \sin(kx - \omega t) \\
p_+ &= Z\omega\xi_m \cos(kx - \omega t) = p_m \cos(kx - \omega t) \\
\dot{p}_+ &= Z\ddot{\xi}_+ = -Z\omega^2 \xi_m \sin(kx - \omega t) = -p_m \omega \sin(kx - \omega t).
\end{aligned}
$$

   For the tidal wave we replace $p$ with $\rho g \eta$, and we can then divide through by $\rho g$ to get:

$$
\begin{aligned}
\xi_+ &= \xi_m \sin(kx - \omega t) \\
\dot{\xi}_+ &= \omega \xi_m \cos(kx - \omega t) \\
\eta_+ &= \eta_m \cos(kx - \omega t) \\
\dot{\eta}_+ &= -\eta_m \omega \sin(kx - \omega t).
\end{aligned}
$$

   At the top of the crest (where the cosine of $(kx - \omega t)$ is equal to one), $\eta_+ = +\eta_m$ (the maximum) $\dot{\eta}_+ = 0$ (momentarily at rest), $\xi_+ = 0$ (the undisturbed horizontal position) and $\dot{\xi}_+ = +\xi_m$, so the particle is moving in the + direction. At the bottom of the trough (where the cosine of $(kx - \omega t)$ is equal to negative one), $\eta_+ = -\eta_m$ (the minimum) $\dot{\eta}_+ = 0$, $\xi_+ = 0$ and $\dot{\xi}_+ = +\xi_m$, so the particle is moving in the + direction.

   Thus the particle describes a clockwise motion when the vertical and horizontal movements are both taken into account.

5. A piston of fixed displacement amplitude and frequency is radiating plane waves into a region filled with an ideal gas. In each of the following cases describe the variation in the power output of the piston as the indicated changes are made.

   (a) The temperature is held constant but the region is pumped out so that the base pressure is lowered. [4]

   **Solution:** For all of these questions, we are really only interested in the radiative power output of the piston since the net convective intensity is zero for any periodic piston motion. For a progressive sinusoidal wave

$$i = p\dot{\xi} = Z\dot{\xi}^2 = \frac{p^2}{Z}.$$

   Additionally, the speed of an acoustic wave is given by $c^2 = \gamma RT_0/M$, so changing the pressure without changing the temperature will not change the value of $c$, but it will change the mass density $\rho_o$.

   For a given displacement amplitude and frequency, we know that

$$p(x,t) = -\mathcal{B}_a \frac{\partial \xi}{\partial x}(x,t),$$

   so for a sinusoidal wave we have

$$\begin{aligned}
\xi(x,t) &= \xi_m \sin(kx - \omega t) \\
\dot{\xi}(x,t) &= \omega \xi_m \cos(kx - \omega t) \\
p(x,t) &= -\mathcal{B}_a k \xi_m \cos(kx - \omega t) \\
&= -\mathcal{B}_a \frac{\omega}{c} \xi_m \cos(kx - \omega t) \\
&= -Z\omega \xi_m \cos(kx - \omega t) \\
&= -\rho_0 c \omega \xi_m \cos(kx - \omega t) \\
\therefore i = p\dot{\xi} &= -\rho_0 c \omega^2 \xi_m^2 \cos^2(kx - \omega t).
\end{aligned}$$

   Thus, lowering the base pressure will lower the density $\rho_o$, and lower the power output of the piston linearly with the decrease in base pressure..

   (b) The pressure remains constant but the temperature is lowered. [3]

   **Solution:** The power output of the piston is again given by $i = -\rho_0 c \omega^2 \xi_m^2 \cos^2(kx - \omega t)$, and the speed by $c^2 = \gamma RT_0/M$, so if the pressure is kept constant but the temperature is lowered, the speed $c$ will decrease while the density $\rho_o$ remains the same and the power output of the piston will decrease proportional to the square root of the decrease in temperature.

   (c) The region is originally filled with nitrogen which is then replaced by hydrogen at the same temperature and pressure. You may need this data for hydrogen and nitrogen: [3]

   hydrogen: $\rho_{H2} = 0.0899\,\mathrm{kg/m^3}$, $c_{H2} = 1270\,\mathrm{m/s}$, $\gamma_{H2} = 1.40$
   nitrogen: $\rho_{N2} = 1.251\,\mathrm{kg/m^3}$, $c_{N2} = 337\,\mathrm{m/s}$, $\gamma_{N2} = 1.40$

   **Solution:** As before using $i = -\rho_0 c \omega^2 \xi_m^2 \cos^2(kx - \omega t)$ we have

$$\begin{aligned}
\frac{i_{H2}}{i_{N2}} &= \frac{\rho_{H2} c_{H2} \omega^2 \xi_m^2 \cos^2(kx - \omega t)}{\rho_{N2} c_{N2} \omega^2 \xi_m^2 \cos^2(kx - \omega t)} \\
&= \frac{\rho_{H2} c_{H2}}{\rho_{N2} c_{N2}} \\
&= \frac{(0.0899)(1270)}{(1.251)(337)} = 0.2708\ldots \\
\frac{i_{H2}}{i_{N2}} &\approx 0.27.
\end{aligned}$$

   Thus, replacing the nitrogen with hydrogen reduces the power output to about 27% of the initial power output.

6. Electromagnetic plane waves. Starting with Maxwell's equations in free space:
$\nabla \cdot \mathbf{E} = 0,\qquad \nabla \cdot \mathbf{H} = 0,\qquad \nabla \times \mathbf{E} = -\mu(\partial \mathbf{H}/\partial t),\qquad \nabla \times \mathbf{H} = \epsilon(\partial \mathbf{E}/\partial t),$

(a) show how a one dimensional wave equation for transverse *magnetic* waves can be developed. [6]

**Solution:** This follows directly from the procedure used in Towne, Chapter 6, examining the case where all the components of $\mathbf{E}$ and $\mathbf{H}$ vanish except for $E_y$ and $H_z$ so Maxwell's equations become

$$\frac{\partial E_y}{\partial z} = 0 \tag{6.01}$$
$$\frac{\partial H_z}{\partial y} = 0 \tag{6.02}$$
$$-\frac{\partial E_y}{\partial z}\hat{\imath} + \frac{\partial E_y}{\partial x}\hat{k} = -\mu\frac{\partial H_z}{\partial t}\hat{k} \tag{6.03}$$
$$\frac{\partial H_z}{\partial y}\hat{\imath} - \frac{\partial H_z}{\partial x}\hat{\jmath} = \epsilon\frac{\partial E_y}{\partial t}\hat{\jmath}. \tag{6.04}$$

Taking the separate components of (6.03) and (6.04) we get

$$\frac{\partial E_y}{\partial y} = 0 \tag{6.05}$$
$$\frac{\partial E_y}{\partial x} = -\mu\frac{\partial H_z}{\partial t} \tag{6.06}$$
$$\frac{\partial H_z}{\partial y} = 0 \tag{6.07}$$
$$-\frac{\partial H_z}{\partial x} = \epsilon\frac{\partial E_y}{\partial t}. \tag{6.08}$$

The equations (6.01), (6.02), (6.05) and (6.07) will only be satisfied if $E_y$ and $H_z$ are functions of at most $x$ and $t$. If we differentiate (6.06) with respect to $t$ and (6.08) with respect to $x$ we get

$$\frac{\partial^2 E_y}{\partial t \partial x} = -\mu\frac{\partial^2 H_z}{\partial t^2} \tag{6.09}$$
$$-\frac{\partial^2 H_z}{\partial x^2} = \epsilon\frac{\partial^2 E_y}{\partial x \partial t}. \tag{6.10}$$

Since the order of differentiating ($t$ or $x$) does not matter, we can put (6.10) into (6.09) to get

$$\frac{\partial^2 H_z}{\partial x^2} = \epsilon\mu\frac{\partial^2 H_z}{\partial t^2}. \tag{6.11}$$

We have that (6.11) is a one dimensional wave equation for transverse *magnetic* waves.Similarly, we could differentiate (6.06) with respect to $x$ and (6.08) with respect to $t$ and put the results together to arrive at the wave equation for electric field waves.

(b) What is the speed of this wave? [2]

**Solution:** Equation (6.11) is a wave equation with $c^2 = 1/\mu\epsilon$.

(c) What type of polarization does this wave have? [2]

**Solution:** This situation is one where the electric and magnetic waves propagate in either direction along the $|\hat{\jmath} \times \hat{k}| = \hat{\imath}$, or $x$, direction with the electric field only having $y$ components, and the magnetic field only having $z$ components. The direction of propagation not only follows from the direction of $\mathbf{S} = \mathbf{E} \times \mathbf{H}$, but also from the fact that both functions are only functions of only $x$ and $t$. Without knowing the functional form of the $x$ and $t$ dependance for $\mathbf{E}$ and $\mathbf{H}$, we cannot tell if the waves are going in the positive, negative, or both directions. Thus this is a linearly polarized plane electromagnetic wave travelling in the $\pm x$ direction.