Physics 380H - Wave Theory | **Homework #11 - Solutions**
Fall 2004 | Due 12:01 PM, Monday 2004/12/06

[35 points total]

"Journal" questions:
– How did the expectation for the course match with how the course actually went? Did you meet your own goals for the course? Did your goals or expectations for the course change through the semester? In what ways?
– Any comments about this week's activities? Course content? Assignment? Lab?

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1. (From Towne P15-5, pg 373) The initial conditions for a string with two fixed ends are

$$y(x,0) = 0, \qquad \text{and} \qquad \dot{y}(x,0) = \sin(2\pi x/l).$$

It is clear that these initial conditions correspond to a particular choice of amplitude and phase of the first harmonic. Show that the formal machinery of the normal-modes expansion leads to the conclusion that this is the only participating mode and exhibit the resulting solution for $y(x,t)$. [10]

Solution: The normal-modes expansion gives a general solution of the form

$$\begin{aligned} y(x,t) = \sum_{n=1}^{\infty} y_n(x,t) &= \sum_{n=1}^{\infty} \{\sin(k_n x)\,[a_n \sin(\omega_n t) + b_n \cos(\omega_n t)]\} \\ &= \sum_{n=1}^{\infty} C_n \sin(k_n x) \sin(\omega_n t + \phi_n), \end{aligned}$$

with $k_n = n\pi/l$ and $\omega_n = ck_n = cn\pi/l$, so we need to find a_n and b_n (or equivalently C_n and ϕ_n).

$$\begin{aligned} b_n &= \frac{2}{l}\int_0^l y(x,0)\sin(k_n x)\,\mathrm{d}x \\ &= \frac{2}{l}\int_0^l (0)\sin(k_n x)\,\mathrm{d}x \\ &= 0 \end{aligned}$$

$$\begin{aligned} a_n\omega_n &= \frac{2}{l}\int_0^l \dot{y}(x,0)\sin(k_n x)\,\mathrm{d}x \\ &= \frac{2}{l}\int_0^l \sin\left(\frac{2\pi x}{l}\right)\sin(k_n x)\,\mathrm{d}x \\ &= \frac{2}{l}\frac{l}{2} = 1 \quad \text{for} \quad m = 2 \\ &= 0 \quad \text{for} \quad m \neq 2. \end{aligned}$$

Thus all a_n and b_n are zero except for $a_2 = 1/(\omega_2) = (1)/(2\omega_1) = 1/(2ck_1) = l/(2\pi c)$. Our solution is thus

$$\begin{aligned} y(x,t) &= a_2 \sin(k_2 x)\sin(\omega_2 t) \\ y(x,t) &= \frac{1}{\omega_2}\sin(k_2 x)\sin(\omega_2 t) \\ &= \frac{1}{2\omega_1}\sin(2k_1 x)\sin(2\omega_1 t) \\ y(x,t) &= \frac{l}{2\pi c}\sin\left(\frac{2\pi}{l}x\right)\sin\left(\frac{2\pi c}{l}t\right). \end{aligned}$$

Note that the given solution satisfies the initial conditions, and that of the infinite series, only the $n = 2$ term is non-zero.

2. (From Towne P15-7, pg 373) A string with two fixed ends is plucked at the centre. Assume that the string is of length l and is at rest at $t = 0$ and that the initial profile is triangular of height h:

$$y(x,0) = \begin{cases} \frac{2h}{l}x, & x \leq \frac{l}{2}, \\ \frac{2h}{l}(l-x), & \frac{l}{2} > x. \end{cases}.$$

Show that the even harmonics will be missing and that the expansion coefficients are $a_n = 0$ and

$$b_n = (-1)^m \frac{8h}{(2m+1)^2\pi^2}$$

for $n = 2m + 1$. [10]

Solution: The initial conditions are $y(x, 0)$ as above and $\dot{y}(x, 0) = 0$. The normal-modes expansion gives a general solution of the form

$$y(x,t) = \sum_{n=1}^{\infty} y_n(x,t) = \sum_{n=1}^{\infty} \{\sin(k_n x)\left[a_n \sin(\omega_n t) + b_n \cos(\omega_n t)\right]\}$$

with $k_n = n\pi/l$ and $\omega_n = ck_n = cn\pi/l$. We need to find a_n and b_n.

$$a_n\omega_n = \frac{2}{l}\int_0^l \dot{y}(x,0)\sin(k_n x)\,\mathrm{d}x = \frac{2}{l}\int_0^l (0)\sin(k_n x)\,\mathrm{d}x = 0$$
$$\therefore a_n = 0$$

$$\begin{aligned}
b_n &= \frac{2}{l}\int_0^l y(x,0)\sin(k_n x)\,\mathrm{d}x \\
&= \frac{2}{l}\int_0^{l/2} y(x,0)\sin(k_n x)\,\mathrm{d}x + \frac{2}{l}\int_{l/2}^{l} y(x,0)\sin(k_n x)\,\mathrm{d}x \\
&= \frac{2}{l}\int_0^{l/2} \frac{2h}{l}x\sin(k_n x)\,\mathrm{d}x + \frac{2}{l}\int_{l/2}^{l} \frac{2h}{l}(l-x)\sin(k_n x)\,\mathrm{d}x \\
&= \frac{4h}{l^2}\left(\int_0^{l/2} x\sin(k_n x)\,\mathrm{d}x + \int_{l/2}^{l}(l-x)\sin(k_n x)\,\mathrm{d}x\right) \\
&= \frac{4h}{l^2}\left(\int_0^{l/2} x\sin(k_n x)\,\mathrm{d}x + l\int_{l/2}^{l}\sin(k_n x)\,\mathrm{d}x - \int_{l/2}^{l} x\sin(k_n x)\,\mathrm{d}x\right) \\
&= \frac{4h}{l^2}\left(\left[\frac{\sin(k_n x)}{k_n^2} - \frac{x\cos(k_n x)}{k_n}\right]_0^{l/2} + l\left[\frac{-\cos(k_n x)}{k_n}\right]_{l/2}^{l} - \left[\frac{\sin(k_n x)}{k_n^2} - \frac{x\cos(k_n x)}{k_n}\right]_{l/2}^{l}\right)
\end{aligned}$$

Evaluating the terms in brackets along with $k_n = n\pi/l$ gives us:

$$
\begin{aligned}
b_n = \frac{4h}{l^2} \Bigg(&\left[\frac{\sin(n\pi/2)}{k_n^2} - \frac{l/2\cos(n\pi/2)}{k_n}\right] + l\left[\frac{-\cos(n\pi)}{k_n}\right] - \left[\frac{\sin(n\pi)}{k_n^2} - \frac{l\cos(n\pi)}{k_n}\right] \\
&- \left[\frac{\sin(k_n 0)}{k_n^2} - \frac{(0)\cos(k_n 0)}{k_n}\right] - l\left[\frac{-\cos(n\pi/2)}{k_n}\right] + \left[\frac{\sin(n\pi/2)}{k_n^2} - \frac{l/2\cos(n\pi/2)}{k_n}\right]\Bigg) \\
= \frac{4h}{l^2} \Bigg(&\left[\frac{\sin(n\pi/2)}{k_n^2} - \frac{l/2\cos(n\pi/2)}{k_n}\right] + l\left[\frac{-\cos(n\pi)}{k_n}\right] - \left[-\frac{l\cos(n\pi)}{k_n}\right] \\
&- l\left[\frac{-\cos(n\pi/2)}{k_n}\right] + \left[\frac{\sin(n\pi/2)}{k_n^2} - \frac{l/2\cos(n\pi/2)}{k_n}\right]\Bigg) \\
= \frac{4h}{l^2} &\left(2\frac{\sin(n\pi/2)}{k_n^2} - 2\frac{l/2\cos(n\pi/2)}{k_n} + l\frac{\cos(n\pi/2)}{k_n}\right) \\
= \frac{4h}{l^2 k_n^2} &\left(2\sin(n\pi/2) - lk_n\cos(n\pi/2) + lk_n\cos(n\pi/2)\right) \\
= \frac{8h}{l^2 k_n^2} &\sin\left(\frac{n\pi}{2}\right) = \frac{8h}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)
\end{aligned}
$$

For even values of n, $\sin(n\pi/2) = 0$ and $b_n = 0$. For odd values of n $\sin(n\pi/2) = \pm 1$. Thus we have that for the odd values of n given by $n = 2m + 1$ for $m = 0, 1, 2, \ldots$ we have that

$$b_n = (-1)^m \frac{8h}{n^2\pi^2} = (-1)^m \frac{8h}{(2m+1)^2\pi^2}.$$

Since all of the even coefficients (a_n as well as b_n) are zero, the overall function is odd. The only odd coefficients are b_n as given above, since $a_n = 0$. All together, the coefficients are:

$$
\begin{aligned}
a_n &= 0, \qquad n = 1, 2, 3, \ldots \\
b_n &= \begin{cases} 0, & n \text{ even} \\ (-1)^m \dfrac{8h}{(2m+1)^2\pi^2}, & n = 2m+1 \text{ odd} \end{cases}
\end{aligned}
$$

3. (From Towne P15-10, pg 374) A string of length l is fixed at both ends. If all points on the string are initially at rest and the initial shape of the string is specified by $y(x,0) = x(\sin kx)$, where $k = \pi/l$,

   (a) find the coefficients in the Fourier series representation of this function. [5]

   Solution: The initial conditions are $y(x,0) = x\sin(\pi x/l)$ as above and $\dot{y}(x,0) = 0$. The normal-modes expansion gives a general solution of the form

$$y(x,t) = \sum_{n=1}^{\infty} y_n(x,t) = \sum_{n=1}^{\infty} \{\sin(k_n x)\,[a_n \sin(\omega_n t) + b_n \cos(\omega_n t)]\}$$

with $k_n = n\pi/l$ and $\omega_n = ck_n = cn\pi/l$. We need to find a_n and b_n.

$$a_n\omega_n = \frac{2}{l}\int_0^l \dot{y}(x,0)\sin(k_n x)\,\mathrm{d}x = \frac{2}{l}\int_0^l (0)\sin(k_n x)\,\mathrm{d}x = 0$$
$$\therefore a_n = 0$$

$$\begin{aligned} b_n &= \frac{2}{l}\int_0^l y(x,0)\sin(k_n x)\,\mathrm{d}x \\ &= \frac{2}{l}\int_0^l x\sin\left(\frac{\pi x}{l}\right)\sin\left(\frac{n\pi x}{l}\right)\mathrm{d}x, \end{aligned}$$

For $n = 1$ we have

$$\begin{aligned} b_1 &= \frac{2}{l}\int_0^l x\sin^2\left(\frac{\pi x}{l}\right)\mathrm{d}x \\ &= \frac{2}{l}\left[\frac{x^2}{4} - \frac{xl}{4\pi}\sin\left(\frac{2\pi x}{l}\right) - \frac{l^2}{8\pi^2}\cos\left(\frac{2\pi x}{l}\right)\right]_0^l \\ &= \frac{2}{l}\left[\frac{l^2}{4} - \frac{ll}{4\pi}\sin\left(\frac{2\pi l}{l}\right) - \frac{l^2}{8\pi^2}\cos\left(\frac{2\pi l}{l}\right) + \frac{l^2}{8\pi^2}\cos(0)\right] \\ &= \frac{2}{l}\left[\frac{l^2}{4}\right] = \frac{l}{2}. \end{aligned}$$

For $n > 1$ we have

$$\begin{aligned} b_n &= \frac{2}{l}\int_0^l \frac{x}{2}\left[\cos\left(\frac{n\pi x}{l} - \frac{\pi x}{l}\right) - \cos\left(\frac{n\pi x}{l} + \frac{\pi x}{l}\right)\right]\mathrm{d}x \\ &= \int_0^l \frac{x}{l}\left[\cos\left((n-1)\frac{\pi x}{l}\right) - \cos\left((n+1)\frac{\pi x}{l}\right)\right]\mathrm{d}x \\ &= \int_0^l \frac{x}{l}\cos\left((n-1)\frac{\pi x}{l}\right)\mathrm{d}x - \int_0^l \frac{x}{l}\cos\left((n+1)\frac{\pi x}{l}\right)\mathrm{d}x \\ &= \left[\frac{l}{(n-1)^2\pi^2}\cos\left((n-1)\frac{\pi x}{l}\right) + \frac{x}{(n-1)\pi}\sin\left((n-1)\frac{\pi x}{l}\right)\right]_0^l \\ &\quad - \left[\frac{l}{(n+1)^2\pi^2}\cos\left((n+1)\frac{\pi x}{l}\right) + \frac{x}{(n+1)\pi}\sin\left((n+1)\frac{\pi x}{l}\right)\right]_0^l \\ &= \frac{l}{(n-1)^2\pi^2}\cos((n-1)\pi) - \frac{l}{(n-1)^2\pi^2} - \frac{l}{(n+1)^2\pi^2}\cos((n+1)\pi) + \frac{l}{(n+1)^2\pi^2} \\ &= \frac{l(\cos((n-1)\pi) - 1)}{(n-1)^2\pi^2} - \frac{l(\cos((n+1)\pi) - 1)}{(n+1)^2\pi^2} \end{aligned}$$

For odd values of n, $\cos((n-1)\pi) = \cos((n+1)\pi) = +1$, thus

$$\begin{aligned} b_n &= \frac{l(1-1)}{(n-1)^2\pi^2} - \frac{l(1-1)}{(n+1)^2\pi^2} \\ &= 0 \end{aligned}$$

For even values of n, $\cos((n-1)\pi) = \cos((n+1)\pi) = -1$, thus

$$\begin{aligned} b_n &= \frac{l(-1-1)}{(n-1)^2\pi^2} - \frac{l(-1-1)}{(n+1)^2\pi^2} \\ &= \frac{-2l}{\pi^2}\left(\frac{1}{(n-1)^2} - \frac{1}{(n+1)^2}\right) \\ &= \frac{-2l}{\pi^2}\frac{(n+1)^2-(n-1)^2}{(n-1)^2(n+1)^2} \\ &= \frac{-2l}{\pi^2}\frac{(n^2+2n+1)-(n^2-2n+1)}{[(n-1)(n+1)]^2} \\ &= \frac{-2l}{\pi^2}\frac{4n}{(n^2-1)^2} \\ &= \frac{-8nl}{\pi^2(n^2-1)^2} \end{aligned}$$

All together, the coefficients are:

$$\begin{aligned} a_n &= 0, \qquad n = 1, 2, 3, \ldots \\ b_n &= \begin{cases} \frac{l}{2}, & n = 1, \\ 0, & n > 1 \text{ and } n \text{ odd} \\ \frac{-8nl}{\pi^2(n^2-1)^2}, & n \text{ even} \end{cases} \end{aligned}$$

(b) Draw a bar graph indicating the relative energies of the first few modes. [5]

Solution: The total energy from the n^{th} mode is given by

$$\mathcal{E}_n = \frac{\sigma l \omega_n^2}{4}(a_n^2 + b_n^2),$$

so in this case since $a_n = 0$ for all n, and $\omega_n = ck_n = cn\pi/l$, we have

$$\mathcal{E}_n = \frac{\sigma l c^2 n^2 \pi^2}{4l^2} b_n^2 = \frac{\sigma c^2 \pi^2}{4l} n^2 b_n^2.$$

For the first few values of n we get the following (define $\mathcal{E}_0 = (\sigma c^2 l)/(\pi^2)$ to make things prettier):

$$\begin{aligned}
\mathcal{E}_1 &= \frac{\sigma c^2 \pi^2}{4l} 1^2 b_1^2 = \frac{\sigma c^2 \pi^2}{4l} \frac{l^2}{4} = \frac{\sigma c^2 l}{\pi^2}\left(\frac{\pi^4}{16}\right) \approx \mathcal{E}_0(6.088\ldots) \\
\mathcal{E}_2 &= \frac{\sigma c^2 \pi^2}{4l} 2^2 b_2^2 = \mathcal{E}_0 \frac{16 \cdot 2^4}{(2^2-1)^4} = \mathcal{E}_0\left(\frac{256}{81}\right) \approx \mathcal{E}_0(3.160\ldots) \\
\mathcal{E}_3 &= 0 \\
\mathcal{E}_4 &= \frac{\sigma c^2 \pi^2}{4l} 4^2 b_4^2 = \mathcal{E}_0 \frac{16 \cdot 4^4}{(4^2-1)^4} = \mathcal{E}_0\left(\frac{4096}{50625}\right) \approx \mathcal{E}_0(0.0809\ldots) \\
\mathcal{E}_5 &= 0 \\
\mathcal{E}_6 &= \frac{\sigma c^2 \pi^2}{4l} 6^2 b_6^2 = \mathcal{E}_0 \frac{16 \cdot 6^4}{(6^2-1)^4} = \mathcal{E}_0\left(\frac{20736}{1500625}\right) \approx \mathcal{E}_0(0.0138\ldots) \\
&\ldots \\
\mathcal{E}_n &= \frac{\sigma c^2 \pi^2}{4l} n^2 b_n^2 = \frac{\sigma c^2 \pi^2}{4l} \frac{64 n^4 l^2}{\pi^4 (n^2-1)^4} = \mathcal{E}_0 \frac{16 n^4}{(n^2-1)^4}, \qquad n \text{ even.}
\end{aligned}$$

We can make a bar graph in units of $\mathcal{E}_0$, with only $\mathcal{E}_1$ and even values of $\mathcal{E}_n$ being non-zero.

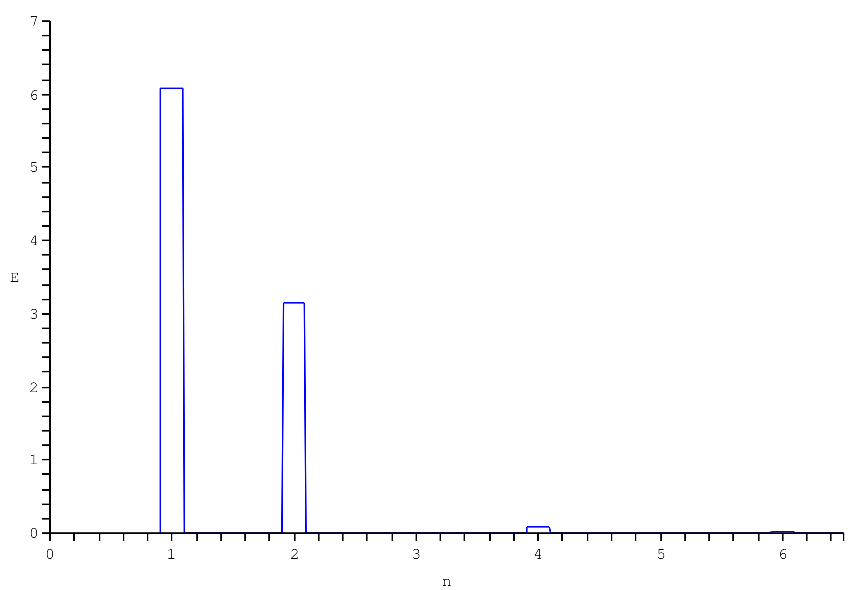

Figure 1: Relative $\mathcal{E}_n$ for $n \leq 6$ in units of $\mathcal{E}_0$.

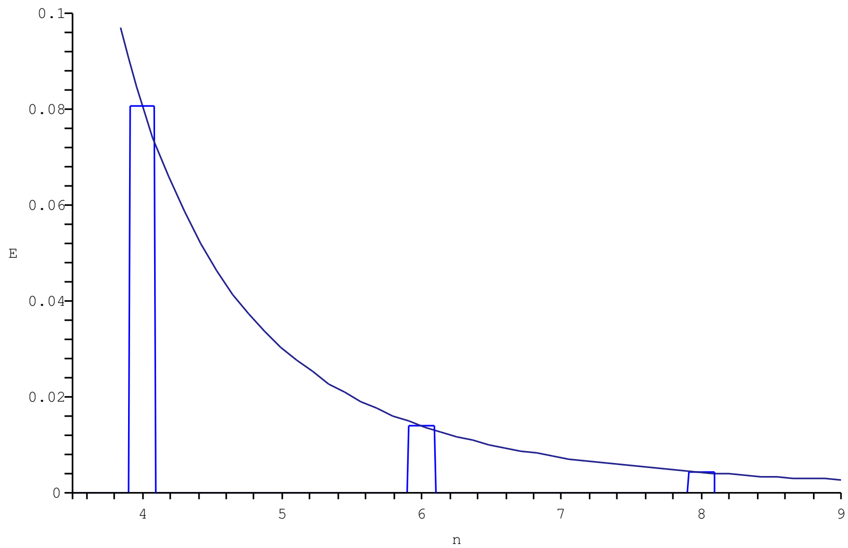

Figure 2: Relative $\mathcal{E}_n$ for $n = 4, 6, 8$ in units of $\mathcal{E}_0$.

We get figure 1 and figure 2 using Maple and the following commands, creating a bar graph by plotting individual points and joining them in a line plot. Note that figure 2 also includes the curve which the non-zero values of $\mathcal{E}_n$ are below for $n > 1$.

```
> restart; with(plots);
> E_0 := 1;
> Efr := n -> E_0*(16*n^4)/(n^2-1)^4;
> En := [ [ .9, 0], [ .91, Pi^4/16], [ 1.09, Pi^4/16], [ 1.1, 0],
  [ 1.9, 0], [ 1.91, Efr(2)], [ 2.09, Efr(2)], [ 2.1, 0],
  [ 3.9, 0], [ 3.91, Efr(4)], [ 4.09, Efr(4)], [ 4.1, 0],
  [ 5.9, 0], [ 5.91, Efr(6)], [ 6.09, Efr(6)], [ 6.1, 0],
  [ 7.9, 0], [ 7.91, Efr(8)], [ 8.09, Efr(8)], [ 8.1, 0] ];
> plot([En], n=0..6.5, E=0..(7));
> plot([En,Efr(n)], n=(3.5)..9, E=0..(0.1));
```

(c) Graph the function $y(x, 0)$ and compare this with a graph of the sum of the *two* most prominent modes in the Fourier series at $t = 0$. [5]

Solution: The two most prominent modes in the Fourier series are $n = 1$ and $n = 2$, so we can graph $y(x, 0)$ as well as $y_1(x, 0) + y_2(x, 0)$ and we expect them to be very similar, particularly since such a small fraction of the energy is in any of the other modes.

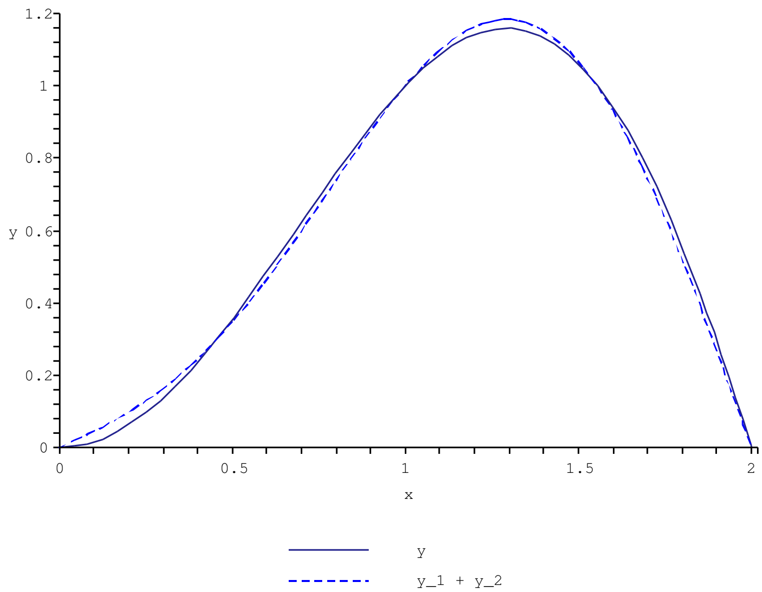

Figure 3: $y(x,0)$ and $y_1(x,0)+y_2(x,0)$

We get figure 3 using Maple and the following commands. We have arbitrarily used a string length of 2 units - any other value would give similar results. Note how close the two curves match, indicating that the two modes used do in fact give a very good approximation of the function.

```
> restart; with(plots);
> l := 2; k := Pi/l; k_1 := Pi/l; k_2 := 2*Pi/l;
> b_1 := l/2; b_2 := (-16*l)/(Pi^2*9);
> ytot := x-> x*sin(k*x);
> y_1 := x-> b_1*sin(k_1*x);
> y_2 := x-> b_2*sin(k_2*x);
> plot([ytot(x), (y_1(x)+y_2(x))], x=0..l*1.01, y=0..1.2, colour=[navy,blue],
  legend=["y", "y_1 + y_2"], linestyle=[1,3]);
```

Headstart for next week, Week 12, starting Monday 2004/12/06:
– Read Chapter 15 “Waves Confined to a Limited Region” in Towne, omit 15-14, 15-15
– – Section 15-12 “Forced motion of a string”
– – Section 15-13 “Eigenfrequencies as resonance frequencies of a string driven sinusoidally at one end”
– – Section 15-16 “Normal modes of a uniformly stretched rectangular membrane”
– – Section 15-17 “Fourier integral analysis on a semi-infinite string”
– – Section 15-18 “Fourier analysis over the whole x-axis”
– Review notes, review texts, review assignments, learn material, do well on exam