Physics 380H - Wave Theory — **Homework #06 - Solutions**

Fall 2004 — Due 12:01 PM, Monday 2004/11/01

[60 points total]

"Journal" questions:
– Approximately how much time per week are you spending on the various aspects of this course, outside of scheduled class times? (ie: lab, assignments, non-assignment pre-reading, general studying, any other categories?) About how much time do you think that you SHOULD be spending on the various aspects of this course? Assuming that you would prefer to reduce your workload, do you have any suggestions on how the course could be arranged to reduce the course workload without significantly reducing the amount and depth of material covered?
– Any comments about this week's activities? Course content? Assignment? Lab?

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1. (From Towne P1-8, pg 17) Given $y(x,t) = \Re\{\mathbf{A}e^{i(\omega t - kx)}\}$, where $\mathbf{A} = Ae^{i\theta}$ is a complex constant. Verify by direct substitution that this function is a solution of the one-dimensional wave equation. What kind of wave is this? Verify that this function conforms with the appropriate relation asserted in (Towne 1-14). [10]

$$\frac{\partial y_\pm}{\partial t} = \mp c \frac{\partial y_\pm}{\partial x} \qquad \text{(Towne 1-14)}$$

Solution: Taking the various derivatives:

$$y(x,t) = \Re\left\{\mathbf{A}e^{i(\omega t - kx)}\right\}$$

$$\frac{\partial y(x,t)}{\partial x} = \Re\left\{-ik\mathbf{A}e^{i(\omega t - kx)}\right\} = -k\Re\left\{i\mathbf{A}e^{i(\omega t - kx)}\right\} \qquad (1.1)$$

$$\frac{\partial^2 y(x,t)}{\partial x^2} = \Re\left\{i^2k^2\mathbf{A}e^{i(\omega t - kx)}\right\} = -k^2 y(x,t) \qquad (1.2)$$

$$\frac{\partial y(x,t)}{\partial t} = \Re\left\{i\omega\mathbf{A}e^{i(\omega t - kx)}\right\} = \omega\Re\left\{i\mathbf{A}e^{i(\omega t - kx)}\right\} \qquad (1.3)$$

$$\frac{\partial^2 y(x,t)}{\partial t^2} = \Re\left\{i^2\omega^2\mathbf{A}e^{i(\omega t - kx)}\right\} = -\omega^2 y(x,t) \qquad (1.4)$$

Multiplying (1.2) by c^2 and setting that equal to (1.4) gives us the wave equation for $ck = \omega$, namely

$$ck = \omega \quad \Longrightarrow \quad \frac{\partial^2 y(x,t)}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y(x,t)}{\partial t^2}.$$

$y(x,t)$ is a transverse wave propagating in the $+x$ direction (since it is a function of $(kx - \omega t)$ it moves to the right). The relation (Towne 1-14) can be verified by multiplying (1.3) by c and setting that equal to (1.1), which is again true if and only if $ck = \omega$. So

$$ck = \omega \quad \Longrightarrow \quad \frac{\partial y(x,t)}{\partial x} = -c\frac{\partial y(x,t)}{\partial t},$$

shows that the given by $y(x,t)$ satisfies the +-wave portion of (Towne 1-14).

2. (From Towne P2-4, pg 36) If the gas filling an organ pipe is changed from air to helium, find the change in pitch of the fundamental vibration. Express the result as a frequency ratio and give the nearest musical interval to which this corresponds. [10]

Solution: The length of the organ pipe does not change, so the wavelength fundamental vibration λ remains constant, while the speed of the sound c is altered since $c = \nu/\lambda$.

$$\frac{\nu_{\text{He}}}{c_{\text{He}}} = \lambda = \frac{\nu_{\text{air}}}{c_{\text{air}}} \quad \Longrightarrow \quad \frac{\nu_{\text{He}}}{\nu_{\text{air}}} = \frac{c_{\text{He}}}{c_{\text{air}}} = \frac{(972)}{(331)} = 2.9365\cdots \approx 3.$$

So the frequency of the fundamental vibration increases by a factor of about three. The musical scale is generally based on factors of two, with each factor of two being one octave.

From http://www.proav.de/music/music_theory.html we have: "Today we use the equal temperament, This system is based on the twelfth root of 2. The ratio of frequencies for each half-tone is equal to $2^{1/12}$. Twelve half-tones give a doubling of frequency and all of these half-tones are exactly the same. The main drawback to this equal temperament is that all major thirds are quite a bit off from where they ought to be, roughly 14% of a half-tone."

Taking the $\log_2$ of our result gives us $\log_2 2.9365\ldots = 1.554125$, which is about one and a half octave, or one octave and about 6.6 half-steps or half-tones. Equivalently, one could look at one half of the calculated value: $1.46827\ldots$ which is more than six half-steps and less than seven half-steps.

From the above reference we see that the "nearest musical interval" is nineteen half-steps which is a "perfect 12th == perfect 5th (one octave)"

3. (From Towne P4-14, pg 82) The fundamental vibration of a viloin string is a standing wave having nodes at the two fixed ends, and is described by the fundtion $y(x,t) = A\sin(kx)\cos(\omega t)$, where $k = \pi/l$, l being the length of the string. Find the total instantaneous kinetic and potential energies (integrated over the length of the string) and show that their sum is constant. [10]

Solution: To calculate the kinetic energy density and potential energy density (that is per unit length) we will need to know the derivatives with respect to t and with respect to x. They are:

$$\epsilon_k = \frac{\sigma}{2}\left(\frac{\partial y}{\partial t}\right)^2 \qquad \frac{\partial y(x,t)}{\partial t} = -\omega A\sin(kx)\sin(\omega t)$$
$$\epsilon_p = \frac{T}{2}\left(\frac{\partial y}{\partial x}\right)^2 \qquad \frac{\partial y(x,t)}{\partial x} = kA\cos(kx)\cos(\omega t)$$

Integrating over the length of the string we have:

$$E_k = \int_0^l \epsilon_k \,\mathrm{d}x = \int_0^l \frac{\sigma}{2}\left(\frac{\partial y}{\partial t}\right)^2 \mathrm{d}x = \frac{\sigma\omega^2A^2}{2}\sin^2(\omega t)\int_0^l \sin^2(kx)\,\mathrm{d}x$$
$$E_p = \int_0^l \epsilon_p \,\mathrm{d}x = \int_0^l \frac{T}{2}\left(\frac{\partial y}{\partial x}\right)^2 \mathrm{d}x = \frac{Tk^2A^2}{2}\cos^2(\omega t)\int_0^l \cos^2(kx)\,\mathrm{d}x$$

Making the substitution of $u = kx$ and $\mathrm{d}u = k\,\mathrm{d}x$ with $kl = \pi$, we have:

$$E_k = \frac{\sigma\omega^2A^2}{2k}\sin^2(\omega t)\int_0^\pi \sin^2(u)\,\mathrm{d}u = \frac{\sigma\omega^2A^2}{2k}\sin^2(\omega t)\left[\frac{u}{2} - \frac{\sin(2u)}{4}\right]_0^\pi = \frac{\pi\sigma\omega^2A^2}{4k}\sin^2(\omega t)$$
$$E_p = \frac{Tk^2A^2}{2k}\cos^2(\omega t)\int_0^\pi \cos^2(u)\,\mathrm{d}u = \frac{Tk^2A^2}{2k}\cos^2(\omega t)\left[\frac{u}{2} + \frac{\sin(2u)}{4}\right]_0^\pi = \frac{\pi Tk^2A^2}{4k}\cos^2(\omega t)$$

We also know that the wave speed $c = \omega/k$ is related to the mass density σ and the string tension T by $\sigma\omega^2 = k^2 T$, so

$$\begin{aligned} E_{tot} = E_k + E_p &= \frac{\pi T k^2 A^2}{4k}\sin^2(\omega t) + \frac{\pi T k^2 A^2}{4k}\cos^2(\omega t) \\ &= \frac{\pi T k^2 A^2}{4k}\left(\sin^2(\omega t) + \cos^2(\omega t)\right) \\ &= \frac{\pi T k A^2}{4} = \frac{l T k^2 A^2}{4} = \frac{\pi\sigma\omega^2 A^2}{4k} = \frac{l\sigma\omega^2 A^2}{4}. \end{aligned}$$

Thus the total energy (kinetic plus potential) integrated over the length of the string is a constant.

4. What surprising consequence did Maxwell's equations provide? How are the permittivity and permeability of free space related to light? Limit your discussion to about 50 words or so. [10]

   Solution: Combining Maxwell's equations in the manner of Towne Chapter 6 gives the result that in free space they permit solutions which are travelling waves of speed $c^2 = 1/\mu\epsilon$. The surprising thing is that the speed of these waves was the same as the measured speed of light. Neither the permittivity nor permeability parameters had ever been associated in any way with light, nor for that matter had any electromagnetic effects ever been associated with optics. Maxwell's equations not only unite electric and magnetic effects with each other, they also in a sense encompass the entire field of optics/photonics as being a sub-field of electromagnetism.

5. Why does monochromatic "natural light" not exhibit the phenomenon of polarization while all sinusoidal waves are intrinsically polarized? Limit your discussion to about 50 words or so. [10]

   Solution: As discussed in Towne in section 7.7, "natural light" is generally not polarized, even though any light source can be thought of as producing various combinations of sinusoidal waves, which certainly are polarized. There are two general arguments to "explain" this fact. One is based on the symmetries and preferred directions associated with a variety of light sources. Since most natural sources do not have any particular structure that would give rise to a preferred direction, each bit of radiation would be expected to have a random polarization, and since the waves do not influence each other, the overall effect would be to have non-polarized light as a result. If think of the various individual sources of light (each atomic oscillator, etc.) that make up any natural light source to be completely uncorrelated in phase, we get the result that the resulting sum of their electromagnetic waves can have a drifting phase, even if it has a relatively well defined frequency/wavelength. Since the typical periods of oscillation of a visible light wave are so very small, we could have overall phase changes that are very "slow" in comparison to the period of oscillation, and yet still be much too quick for any but the most sensitive instruments to detect.

6. (From Towne P7-14, pg 133) The channeled spectrum obtained by reflection from a thin film surrounded by air contains only two dark bands, centered at 4500 Å and 6000 Å. What is the optical thickness of the film? ("Optical thickness" is a term used for the product of the index of refraction and the geometrical thickness. Neglect the dependence of index of refraction on wavelength.) [10]

   Solution: Since the film has an identical substance (air) on each side of it, we know that reflection from one surface will result in a phase change of π, while reflection from the other side of the film will result in no phase change. We do not know which surface has the phase change for at least in principle, the film could have a lower index of refraction than the air, even though that is unlikely. Thus if the total optical path difference $\delta = 2nt$ (t is the physical thickness of the film and n is the relative index of refraction between the film and air. Note that nt is the "optical thickness" that we are trying to find.) between light reflecting off of the front surface versus light reflecting off of the back surface is an integral number of wavelengths of the light, there will be *destructive* interference and we will observe a dark band. Since we observe destructive interference for the two given wavelengths, but no other wavelengths between these two, the integer corresponding with one wavelength of light must be one greater (or lesser) than the integer corresponding with the other wavelength.

   Given $\lambda_1 = 4500\,\text{Å}$ and $\lambda_2 = 6000\,\text{Å}$ and $m_1 = m_2 + 1$, we have:

$$\delta = 2nt = m_1\lambda_1 = m_2\lambda_2 \quad \Longrightarrow \quad \frac{m_1}{m_2} = \frac{\lambda_2}{\lambda_1} = \frac{4}{5} \quad \Longrightarrow \quad m_1 = 4, \qquad m_2 = 5$$

$$\therefore nt = \frac{m_1\lambda_1}{2} = \frac{m_2\lambda_2}{2} = 9000\,\text{Å}.$$

   The optical thickness of the film is $nt = 9000\,\text{Å}$.

Headstart for next week, Week 07, starting Monday 2004/11/01:
– Read Chapter 11 "Interference Pattern from a Pair of Point Sources" in Towne, omit 11-8 through 11-15
– – Section 11-1 "Introduction"
– – Section 11-2 "Sources close together compared with a wavelength; the dipole source"
– – Section 11-3 "Various interference patters for $d \sim \lambda$"
– – Section 11-4 "Total power radiated from a pair of point sources"
– – Section 11-5 "The phenomenon of beats"
– – Section 11-6 "Interference patterns when $kd \gg 1$"
– – Section 11-7 "Young's experiment"