Physics 380H - Wave Theory **Homework #04 - Solutions**
Fall 2004 Due 12:01 PM, Tuesday 2004/10/12

[30 points total]

"Journal" questions:
– Give an example of a time you made use of physics knowledge you gained from a physics course, outside of schoolwork. What physics phenomena have you noticed outside of the classroom? Have you noticed or made use of *waves* outside of class? In what context?
– Any comments about this week's activities? Course content? Assignment? Lab?

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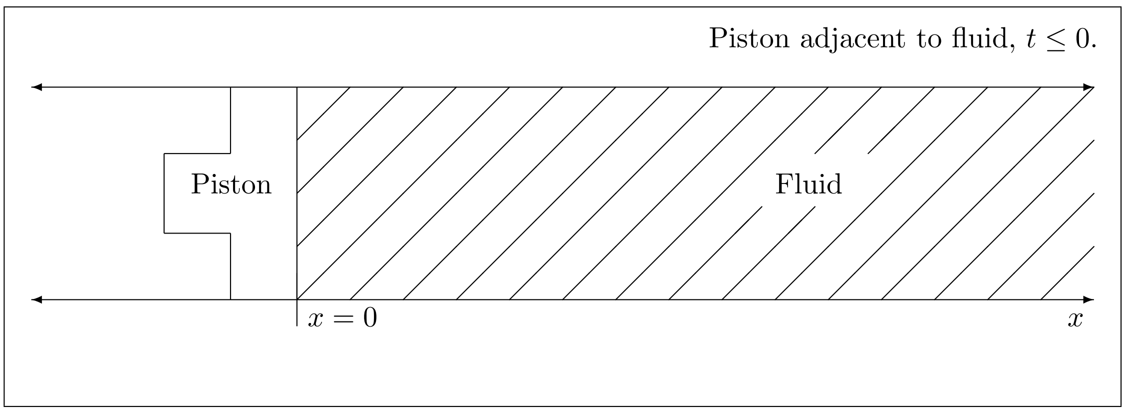
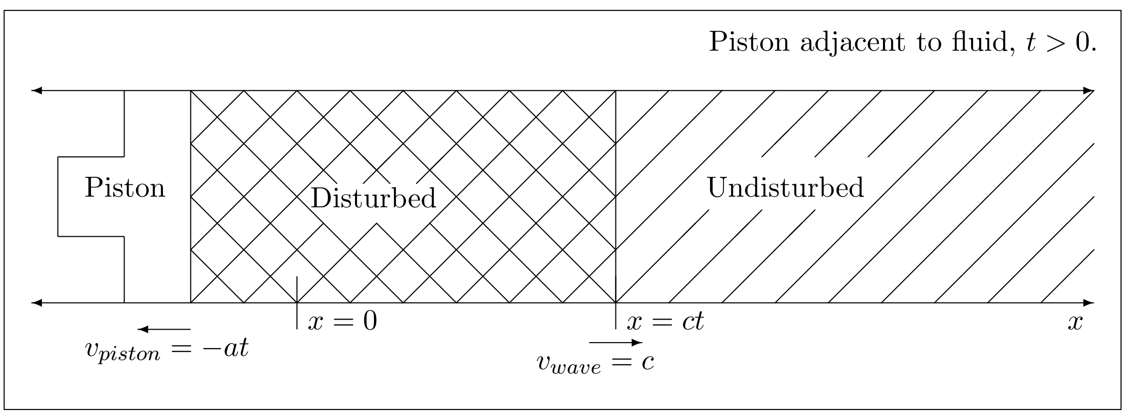
1. (From Towne P3-4, pg 60) A piston is located at $x = 0$, adjacent to a fluid extending throughout $x > 0$. The piston is initially at rest, and there is no disturbance in the fluid. At $t = 0$ the piston starts to move with constant negative acceleration: $B(t) = -at^2/2$. Describe the condition of the fluid at some later time. [10]

   Solution: Before the piston starts to move, the situation looks like this:





For $t > 0$ the fluid will begin to accelerate to the left, into the $x < 0$ region. Following the work of Towne in Section 3-3, the boundary condition is that the fluid adjacent to the piston must stay in contact with the piston, so $\xi(0, t) = B(t)$ (which assumes that cavitation does not develop). (Note that we are using functions of the form $(t - x/c)$ and $(t + x/c)$ rather than $(x - ct)$ and $(x + ct)$ because it makes the algebra a bit simpler.) Since there are no reflection surfaces for $x > 0$ and no other possible sources of waves coming from the right, there should be no -wave present, so the general form of ξ is given by the standard form with $g = 0$:

$$\begin{aligned}\xi(x, t) &= f(t - x/c) + g(t + x/c) \\ &= f(t - x/c).\end{aligned}$$

The boundary condition gives

$$B(t) = \xi(0,t) = f(t),$$

from which it follows that

$$\xi(x,t) = f(t - x/c) = B(t - x/c).$$

Thus for the case under consideration

$$\xi(x,t) = B(t - x/c) = \frac{-a}{2}\,(t - x/c)^2$$

for $x \leq ct$ and $\xi(x,t) = 0$ for $x > ct$. Note that even though each fluid particle moves to the *left*, the *wave* (ie. the disturbance) moves to the right.

At some later time t, the fluid to the right of position $x > ct$ will still be undisturbed, while the fluid to the left of this point will be moving to the left at speed given by

$$\dot{\xi}(x,t) = \frac{\partial \xi}{\partial t}(x,t) = -a\left(t - x/c\right),$$

and the fluid acceleration is given by

$$\ddot{\xi}(x,t) = \begin{cases} -a, & x < ct, \\ 0, & ct \leq x. \end{cases}$$

Since $p(x,t) = Z\dot{\xi}(x,t)$, the pressure wave is given by:

$$p(x,t) = Z\dot{\xi}(x,t) = -Za\left(t - x/c\right),$$

for $x \leq ct$ and of course $p(x,t) = 0$ for $x > ct$.

We should also note that while the acoustic approximations will hold provided a is small enough, eventually the speed of the piston at will be greater than the speed of wave propagation c and one might think that the acoustic approximation will no longer hold. However, since the fluid is moved along with the piston, locally there should be no problem with at getting very large, as long as no cavitation occurs.

2. (From Towne P3-9, pg 60) Start with the boundary condition of a free surface, $p(0,t) = 0$, and apply this directly (in a manner similar to the method of Section 3-2) to deduce $p(x,t)$ and $\dot{\xi}(x,t)$. [10]

Solution: The free surface boundary condition $p(0,t) = 0$ can be combined with the wave equation for $p(x,t)$ to give us

$$p(x,t) = f(t - x/c) + g(t + x/c) \quad \Longrightarrow \quad p(0,t) = 0 = f(t) + g(t) \quad \Longrightarrow \quad g(t) = -f(t).$$

This tells us that the reflected (pressure) wave is inverted from the incident wave, just as the displacement wave behaves for a rigid surface. Since we want p to be zero (a node) at the surface we choose for the sinusoidal incident wave $f(t) = p_m \cos(\omega t)$ and we get:

$$\begin{aligned} p(x,t) &= f(t - x/c) + g(t + x/c) \\ &= f(t - x/c) - f(t + x/c) \\ &= p_m \cos(\omega(t - x/c)) - p_m \cos(\omega(t + x/c)) \\ &= p_m \cos(\omega t - kx) - p_m \cos(\omega t + kx) \\ &= 2p_m \sin(kx) \sin(\omega t). \end{aligned}$$

Since $p(x,t) = -\mathcal{B}_a \frac{\partial \xi}{\partial x}$, the above expression for $p(x,t)$ allows us to calculate expressions for $\xi(x,t)$ and $\dot{\xi}(x,t)$, by integrating and differentiating, thus

$$\begin{aligned} \xi(x,t) &= \frac{2p_m}{\mathcal{B}_a k} \cos(kx) \sin(\omega t) \\ &= \xi_m \cos(kx) \sin(\omega t) \end{aligned} \qquad \begin{aligned} \dot{\xi}(x,t) &= \frac{2p_m \omega}{\mathcal{B}_a k} \cos(kx) \cos(\omega t) \\ &= \frac{2p_m c}{\mathcal{B}_a} \cos(kx) \cos(\omega t) \\ &= \dot{\xi}_m \cos(kx)\ cos(\omega t). \end{aligned}$$

So there are displacement nodes at half-wavelength intervals, with the first one being at one quarter wavelength. At the surface, the amplitude of the displacement is a maximum.

3. (From Towne P4-1, pg 81) Find the exact expression for the potential energy density W_{pot} of an *ideal gas*. Show that this reduces to the correct form when the acoustic approximation is made. (Note: Since the desired quantity contains second-order terms in s, a more accurate expression than $s = -\partial\xi/\partial x$ is required to show the equivalence.) [10]

Solution: The potential energy (per unit volume) is given by

$$W_{pot} = \rho_0 \int_{\rho_0}^{\rho} P_a(\rho) \frac{\mathrm{d}\rho}{\rho^2} \qquad \text{(see Towne pg 64)},$$

and the equation of state for an ideal gas is

$$P_a(\rho) = P_0 \left(\frac{\rho}{\rho_0}\right)^{\gamma} \qquad \text{(see Towne pg 25)}.$$

We put these two together and we get

$$
\begin{aligned}
W_{pot} &= \rho_0 \int_{\rho_0}^{\rho} P_0 \left(\frac{\rho}{\rho_0}\right)^{\gamma} \frac{\mathrm{d}\rho}{\rho^2} \\
&= \frac{\rho_0 P_0}{\rho_0^{\gamma}} \int_{\rho_0}^{\rho} \rho^{\gamma} \frac{\mathrm{d}\rho}{\rho^2} \\
&= \frac{P_0}{\rho_0^{(\gamma-1)}} \int_{\rho_0}^{\rho} \rho^{(\gamma-2)} \mathrm{d}\rho \\
&= \frac{P_0}{\rho_0^{(\gamma-1)}} \int_{\rho_0}^{\rho} \rho^{(\gamma-2)} \mathrm{d}\rho \\
&= \frac{P_0}{\rho_0^{(\gamma-1)}} \left[\frac{\rho^{(\gamma-1)}}{(\gamma-1)}\right]_{\rho_0}^{\rho} \\
&= \frac{P_0}{(\gamma-1)\rho_0^{(\gamma-1)}} \left[\rho^{(\gamma-1)}\right]_{\rho_0}^{\rho} \\
&= \frac{P_0}{(\gamma-1)\rho_0^{(\gamma-1)}} \left[\rho^{(\gamma-1)} - \rho_0^{(\gamma-1)}\right] \\
W_{pot} &= \frac{P_0}{(\gamma-1)} \left[\left(\frac{\rho}{\rho_0}\right)^{(\gamma-1)} - 1\right]
\end{aligned}
$$

We would like to show that this reduces to Towne's 4-12 from pg 67, namely

$$
W_{pot} = -P_0 \frac{\partial \xi}{\partial x} + \frac{\mathcal{B}_a s^2}{2} = -P_0 \frac{\partial \xi}{\partial x} + \frac{p^2}{2\mathcal{B}_a}
$$

I think that we need to expand the expression for ρ to encompass more than just the linear term, however I cannot seem to go anywhere further with that idea... I cannot figure out what to do with those pesky exponents containing $(\gamma - 1)$...

$$
\begin{aligned}
\rho &= \rho_0 \left(1 + \frac{\partial \xi}{\partial x}\right)^{-1} = \rho_0 \left(1 - \left(\frac{\partial \xi}{\partial x}\right) + \left(\frac{\partial \xi}{\partial x}\right)^2 - \left(\frac{\partial \xi}{\partial x}\right)^3 + \cdots\right) \\
\rho^2 &= \rho_0^2 \left(1 + \frac{\partial \xi}{\partial x}\right)^{-2} = \rho_0^2 \left(1 - 2\left(\frac{\partial \xi}{\partial x}\right) + 3\left(\frac{\partial \xi}{\partial x}\right)^2 - 4\left(\frac{\partial \xi}{\partial x}\right)^3 + \cdots\right) \\
s &= \frac{\rho - \rho_0}{\rho_0} = -\left(\frac{\partial \xi}{\partial x}\right) + \left(\frac{\partial \xi}{\partial x}\right)^2 - \left(\frac{\partial \xi}{\partial x}\right)^3 + \cdots \\
s &= \frac{\rho - \rho_0}{\rho_0} \quad \Longrightarrow \quad \rho = \rho_0(s+1)
\end{aligned}
$$

$$
\begin{aligned}
W_{pot} &= \frac{P_0}{(\gamma-1)} \left[\left(\frac{\rho}{\rho_0}\right)^{(\gamma-1)} - 1\right] \\
&= \frac{P_0}{(\gamma-1)} \left[\left(\frac{\rho_0(s+1)}{\rho_0}\right)^{(\gamma-1)} - 1\right] \\
&= \frac{P_0}{(\gamma-1)} \left[(s+1)^{(\gamma-1)} - 1\right]
\end{aligned}
$$

$$
\begin{aligned}
W_{pot} &= \frac{P_0}{(\gamma-1)}\left[\left(\frac{\rho}{\rho_0}\right)^{(\gamma-1)} - 1\right] \\
&= \frac{P_0}{(\gamma-1)}\left[\left(\frac{\rho_0\left(1+\frac{\partial\xi}{\partial x}\right)^{-1}}{\rho_0}\right)^{(\gamma-1)} - 1\right] \\
&= \frac{P_0}{(\gamma-1)}\left[\left(\left(1+\frac{\partial\xi}{\partial x}\right)^{-1}\right)^{(\gamma-1)} - 1\right] \\
&\approx \frac{P_0}{(\gamma-1)}\left[\left(1-\left(\frac{\partial\xi}{\partial x}\right)+\left(\frac{\partial\xi}{\partial x}\right)^{2}\right)^{(\gamma-1)} - 1\right]
\end{aligned}
$$

Headstart for next week, Week 05, starting Tuesday 2004/10/12:
– Read Chapter 4 “Energy in a Sound Wave; Isomorphisms” in Towne, omit 4-7
– – Section 4-8 “Interference between superposed waves”
– – Section 4-9 “Measurement of intensity in decibels”
– – Section 4-10 “Energy definitions for transverse waves on a string”
– – Section 4-11 “Energy relations for transverse waves on a string; isomorphisms”
– – Section 4-12 “Bountry value problems for transverse waves on a string”
– Read Chapter 6 “The Electromagnetic Plane Wave” in Towne, omit 6-4, 6-5
– – Section 6-1 “Maxwell’s equations”
– – Section 6-2 “A solution to Maxwell’s equations for a special situation”
– – Section 6-3 “Implications of the electromagnetic theory of light”
– – Section 6-6 “A linearly polarized transverse plane wave”
– Read Chapter 7 “Analytical Description of Polarized Electromagnetic Plane Waves” in Towne
– – Section 7-1 “Introduction”
– – Section 7-2 “More complete description of the linearly polarized sinusoidal plane wave”
– – Section 7-3 “Reflection from a dielectric surface obtained by appeal to an isomorphism”
– – Section 7-4 “Reflection from a perfect conductor; direct evidence of standing waves”
– – Section 7-5 “Consideration of a more general sinusoidal plane wave”