Physics 202H - Introductory Quantum Physics I
Homework #06 - Solutions

Fall 2004 Due 5:01 PM, Monday 2004/11/01

[65 points total]

"Journal" questions. Briefly share your thoughts on the following questions:
– About how much time per week are you spending on the various aspects of this course, outside of scheduled class times? (ie: lab, assignments, non-assignment pre-reading, general studying, etc.?) About how much time do you think that you SHOULD be spending on the various aspects of this course? Do you have any suggestions on how the course could be arranged to reduce the course workload without significantly reducing the amount and depth of material covered?
– Any comments about this week's activities? Course content? Assignment? Lab?

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1. (From Eisberg & Resnick, Q 4-9, pg 120) For the Bohr hydrogen atom orbits, the potential energy is negative and greater in magnitude than the kinetic energy. What does this imply? Limit your discussion to about 50 words or so. [10]

   Solution: The value of the potential energy depends on where the zero of potential energy is defined. For electric charge systems, it is usually most convenient to define zero to be at "infinite" distance, thus for an attractive force such as between an atomic nucleus (positive charge) and an electron (negative charge), the potential energy will be negative, since as the two charges come closer together, their potential energy decreases. Since zero potential energy is defined to be the situation when the electron is freed from the nucleus, the kinetic energy of any bound electron must be less in magnitude than the magnitude of the potential energy. If the kinetic energy of the electron was greater than the magnitude of the (negative) potential energy, the electron would have enough kinetic energy to "get away" from the nucleus, and would not be bound to it in an atomic system.

2. (From problem 2-27, "Simple Nature", Crowell, pg 107) Assume that the kinetic energy of an electron the $n = 1$ state of a hydrogen atom is on the same order of magnitude as the absolute value of its total energy, and estimate a typical speed at which it would be moving. (It cannot really have a single, definite speed, because its kinetic and interaction energy trade off at different distances from the proton, but this is just a rough estimate of a typical speed.) Based on this speed, were we justified in assuming that the electron could be described nonrelativistically? [10]

   Solution: For the $n = 1$ state of a hydrogen atom, the binding energy is about $-13.6\,\mathrm{eV}$ from

   $$E = -\left(\frac{1}{4\pi\epsilon_0}\right)^2 \frac{me^4Z^2}{2\hbar^2} = -2.17\times 10^{-18}\,\mathrm{J} = -13.6\,\mathrm{eV}.$$

   For non non-relativistic velocities,

   $$K = \frac{1}{2}mv^2 \quad \Longrightarrow \quad v = \sqrt{\frac{2K}{m}},$$

   so if we assume that the kinetic energy is about the same as the binding energy in magnitude, we get

   $$v \approx \sqrt{\frac{(4\times 10^{-18}\,\mathrm{J})}{(9.109\times 10^{-31}\,\mathrm{kg})}} \approx 2\times 10^6\,\mathrm{m/s}.$$

   Since this is a bit less than 1% of the speed of light (it is close to 0.7% of c), our non-relativistic assumption was justified. Note that we only kept one significant figure in our result since we are really only interested in an order of magnitude estimate of the velocity, based on an order of magnitude estimate of the kinetic energy.

3. (From problem 2-33, "Simple Nature", Crowell, pg 108) A muon is a subatomic particle that acts exactly like an electron except that its mass is 207 times greater. Muons can be created by cosmic rays, and it can happen that one of an atom's electrons is displaced by a muon, forming a muonic atom. If this happens to a hydrogen atom, the resulting system consists simply of a proton plus a muon.

   (a) Based on the results of Crowell Section 2.4.4, how would the size of a muonic hydrogen atom in its ground state compare with the size of the normal atom? [10]

   Solution: The radius of the ground state of the Bohr model is given by

$$
\begin{aligned}
r &= 4\pi\epsilon_0 \frac{n^2\hbar^2}{mZe^2} && \text{Eisberg \& Resnick} \\
&\approx 4\pi\epsilon_0 \frac{n^2h^2}{me^2} = \frac{n^2h^2}{mke^2} && \text{Crowell}
\end{aligned}
$$

   The muon has a much greater mass than the electron, $m_\mu = 207 m_e$, so

$$\frac{r_\mu}{r_e} = \frac{n^2\hbar^2}{m_\mu Ze^2}\frac{m_e Ze^2}{n^2\hbar^2} = \frac{m_e}{m_\mu} = \frac{1}{207} = 0.0048309 \approx 0.5\%.$$

   More accurately, the reduced mass of of the muon, following Eisberg & Resnick section 4-7 with the nuclear mass of $M = 1836 m_e$ is about $\mu_\mu = 186 m_e$, giving $r_\mu \approx r_e/186 \approx 0.0054 r_e$.

   (b) If you were searching for muonic atoms in the sun or in the earth's atmosphere by spectroscopy, in what part of the electromagnetic spectrum would you expect to find the absorption lines? Why? (Hint: See Eisberg & Resnick, P 4-30, pg 122.) [5]

   Solution: The nth energy state of a Bohr atom is given by

$$E_n = -\left(\frac{1}{4\pi\epsilon_0}\right)^2 \frac{me^4Z^2}{2\hbar^2}\left(\frac{1}{n^2}\right).$$

   For hydrogen, the ground state energy is $E_{e-1} = -2.17\times10^{-18}\,\mathrm{J} = -13.6\,\mathrm{eV}$. Increasing the mass of the orbiting particle will proportionately increase the magnitude of each energy state. The shortest wavelength that is possible (for emission) is given by a transition from the $n = \infty$ to the $n = 1$ state. The absorption spectrum is identical to the emission spectrum, the only differences being various negative signs signifying absorption rather than emission.

$$
\begin{aligned}
\Delta E_\mu = E_{\mu-\infty} - E_{\mu-1} &= \left(\frac{1}{4\pi\epsilon_0}\right)^2 \frac{m_\mu e^4}{2\hbar^2}\left(\frac{1}{1^2} - \frac{1}{\infty^2}\right) \\
&= \left(\frac{1}{4\pi\epsilon_0}\right)^2 \frac{m_\mu e^4}{2\hbar^2}\left(\frac{1}{1^2}\right) \\
&= |E_{e-1}|\frac{m_\mu}{m_e} = (13.6\,\mathrm{eV})(207) = (2815.2\,\mathrm{eV}) \\
\therefore \lambda_{\mu\,\mathrm{max}} &= \frac{hc}{\Delta E_\mu} = \frac{hcm_e}{|E_{e-1}|\,m_\mu} = \frac{(4.1356668\ldots\times10^{-15}\,\mathrm{eV\cdot\ s})(299792458\,\mathrm{m/s})}{(2815.2\,\mathrm{eV})} \\
&= (4.404\ldots\times10^{-10}\,\mathrm{m}) \approx 4.40\,\text{Å}
\end{aligned}
$$

   If we had used the reduced mass of $\mu_\mu = 186 m_e$ we would instead have the result of $\lambda_{\mu\,\mathrm{max}} \approx 4.90\,$Å.

   Similarly one could find the $n = 2 \to n = 1$ transition wavelengths (about 6.5 Å or 5.8 Å for the reduced mass value), and the transitions from other energy levels. These transitions are all in the same X-ray region (sometimes called the "hard X-ray" region) of the spectrum, with frequencies greater than that of visible or ultra-violet light.

4. (From Eisberg & Resnick, P 4-33, pg 122) Using Bohr's model, calculate the energy required to remove the electron from singly ionized helium. [5]

   Solution: This is equivalent to finding the ground state energy of singly ionized helium, and taking the absolute value of that. We want to find the energy needed for a transition from the $n = 1$ to the $n = \infty$ state, with $Z = 2$.

$$E_n = -\left(\frac{1}{4\pi\epsilon_0}\right)^2 \frac{me^4Z^2}{2\hbar^2}\left(\frac{1}{n^2}\right).$$

$$\begin{aligned}\Delta E = E_1 - E_\infty &= -\left(\frac{1}{4\pi\epsilon_0}\right)^2 \frac{me^4Z^2}{2\hbar^2}\left(\frac{1}{1^2} - \frac{1}{\infty^2}\right)\\ &= -\left(\frac{1}{4\pi\epsilon_0}\right)^2 \frac{me^4 2^2}{2\hbar^2}\left(\frac{1}{1^2}\right) = -(4)(13.6\,\text{eV})\\ &= -(54.4\,\text{eV}) = -(8.715\ldots \times 10^{-20}\,\text{J})\end{aligned}$$

   It will take about $54.4\,\text{eV}$ or 8.72×10^{-20} J of energy to remove an electron from singly ionized helium.

5. (From Eisberg & Resnick, P 4-35, pg 122) A $3.00\,\text{eV}$ electron is captured by a bare nucleus of helium. If a $2400\,\text{Å}$ photon is emitted, into what level was the electron captured? [5]

   Solution: Energy is conserved in this interaction. The initial energy is just the given kinetic energy of the electron, $E_i = K_i = 3.00\,\text{eV}$, since the potential energy of atom is zero when the electron is far away from the nucleus. The final energy of the system is made up of the energy of the photon $E_{photon} = hc/\lambda$ plus the energy of the electron in the $n = 1$ state in the helium atom ($Z = 2$).

$$E_n = -\left(\frac{1}{4\pi\epsilon_0}\right)^2 \frac{me^4Z^2}{2\hbar^2}\left(\frac{1}{n^2}\right) = -(4)(13.6\,\text{eV})\left(\frac{1}{n^2}\right) = -(54.4\,\text{eV})\left(\frac{1}{n^2}\right)$$

$$E_{photon} = \frac{hc}{\lambda} = \frac{(4.1356668\ldots \times 10^{-15}\,\text{eV}\cdot\ \text{s})(299792458\,\text{m/s})}{(2400 \times 10^{-10}\,\text{m})} = (5.166\ldots\ \text{eV})$$

$$\begin{aligned}E_f &= E_i\\ E_n + E_{photon} &= K_i\\ E_n &= K_i - E_{photon}\\ -(54.4\,\text{eV})\left(\frac{1}{n^2}\right) &= (3.00\,\text{eV}) - (5.166\ldots\ \text{eV})\\ \frac{1}{n^2} &= \frac{(2.166\ldots)}{(54.4)} = 0.039816\ldots\\ n^2 &= 25.11\ldots\\ n &= 5\end{aligned}$$

   The electron was captured into the $n = 5$ level by the bare nucleus of helium.

6. How is the de Broglie wavelength related to Bohr's quantization of angular momentum? In this case what is the periodic function that the Wilson-Sommerfeld rule is applied to? Limit your discussion to about 50 words or so. [10]

   Solution: For an electron in a circular orbit of a Bohr atom, requiring that the orbital angular momentum is quantized by $L = n\hbar$ is equivalent to the requirement that the circumference of the orbit is exactly an integral number of de Broglie wavelengths, $2\pi r = n\lambda$. Each of these conditions implies the other, and each is implied by the other, so stating either one as a postulate is sufficient to have the other one as a result.

   In this case, the periodic function in question is the angle θ of the electron in the orbit, which repeats itself once per orbit. As a function it is a "sawtooth", rising from zero at a constant rate and then dropping back to zero discontinuously every $2\pi\,\mathrm{rad} = 360°$.

7. (From Eisberg & Resnick, P 4-43, pg 123) Assume the angular momentum of the earth of mass $m_\oplus = 6.0 \times 10^{24}\,\mathrm{kg}$ due to its motion around the sun at a radius $r_\oplus = 1.5 \times 10^{11}\,\mathrm{m}$ to be quantized according to Bohr's relation $L = nh/2\pi$. What is the value of the quantum number n? Could such quantization be detected? [10]

   Solution: Since $L = m_\oplus v_\oplus r_\oplus = nh/2\pi$ and $v_\oplus = 2\pi r_\oplus/T$ where T is one year or 31 556 926 seconds, we have

$$\begin{aligned} n &= 2\pi\frac{L}{h} = 2\pi\frac{m_\oplus v_\oplus r_\oplus}{h} = 2\pi\frac{m_\oplus 2\pi r_\oplus r_\oplus}{hT} \\ &= \frac{m_\oplus (2\pi r_\oplus)^2}{hT} \\ &= \frac{(6.0\times 10^{24}\,\mathrm{kg})4\pi^2(1.5\times 10^{11}\,\mathrm{m})^2}{(6.6260\ldots\times 10^{-34}\,\mathrm{J\cdot s})(3.1556\ldots\times 10^{7}\,\mathrm{s})} \\ &= 2.5488\ldots\times 10^{74}. \end{aligned}$$

   This is a very large value for n. In order to differentiate between n and $n+1$, we would need to know the measured values of $m_\oplus$ and $r_\oplus$ and the other parameters to more than 70 digits of accuracy. For example since the diameter of a proton is about $10^{-15}\,\mathrm{m}$, we would need to know $r_\oplus$ to within about $10^{11+15-74} = 10^{-48}$ of the size of a proton! Clearly this is impossible.

Headstart for next week, Week 07, starting Monday 2004/11/01:
– Review Section 2.3.6 "The Schrödinger equation" in "Simple Nature" by Crowellk
– Read Chapter 5 "Schroedinger's Theory of Quantum Mechanics" in Eisberg & Resnick
– – Section 5.1 "Introduction"
– – Section 5.2 "Plausibility Argument Leading to Schroedinger's Equation"
– – Section 5.3 "Born's Interpretation of Wave Functions"