PREFACE TO THE SECOND EDITION

The many developments that have occurred in the physics of quantum systems since the publication of the first edition of this book—particularly in the field of elementary particles—have made apparent the need for a second edition. In preparing it, we solicited suggestions from the instructors that we knew to be using the book in their courses (and also from some that we knew were *not*, in order to determine their objections to the book). The wide acceptance of the first edition made it possible for us to obtain a broad sampling of thought concerning ways to make the second edition more useful. We were not able to act on all the suggestions that were received, because some were in conflict with others or were impossible to carry out for technical reasons. But we certainly did respond to the general consensus of these suggestions.

Many users of the first edition felt that new topics, typically more sophisticated aspects of quantum mechanics such as perturbation theory, should be added to the book. Yet others said that the level of the first edition was well suited to the course they teach and that it should not be changed. We decided to try to satisfy both groups by adding material to the new edition in the form of new appendices, but to do it in such a way as to maintain the decoupling of the appendices and the text that characterized the original edition. The more advanced appendices are well integrated in the text but it is a one-way, not two-way, integration. A student reading one of these appendices will find numerous references to places in the text where the development is motivated and where its results are used. On the other hand, a student who does not read the appendix because he is in a lower level course will not be frustrated by many references in the text to material contained in an appendix he does not use. Instead, he will find only one or two brief parenthetical statements in the text advising him of the existence of an optional appendix that has a bearing on the subject dealt with in the text.

The appendices in the second edition that are new or are significantly changed are: Appendix A, The Special Theory of Relativity (a number of worked-out examples added and an important calculation simplified); Appendix D, Fourier Integral Description of a Wave Group (new); Appendix G, Numerical Solution of the Time-Independent Schroedinger Equation for a Square Well Potential (completely rewritten to include a universal program in BASIC for solving second-order differential equations on microcomputers); Appendix J, Time-Independent Perturbation Theory (new); Appendix K, Time-Dependent Perturbation Theory (new); Appendix L, The Born Approximation (new); Appendix N, Series Solutions of the Angular and Radial Equations for a One-Electron Atom (new); Appendix Q, Crystallography (new); Appendix R, Gauge Invariance in Classical and Quantum Mechanical Electromagnetism (new). Problem sets have been added to the ends of many of the appendices, both old and new. In particular, Appendix A now contains a brief but comprehensive set of problems for use by instructors who begin their "modern physics" course with a treatment of relativity.

A large number of small changes and additions have been made to the text to improve and update it. There are also several quite substantial pieces of new material, including: the new Section 13-8 on electron-positron annihilation in solids; the additions to Section 16-6 on the Mössbauer effect; the extensive modernization of the last half of the introduction to elementary particles in Chapter 17; and the entirely new Chapter 18 treating the developments that have occurred in particle physics since the first edition was written.

We were very fortunate to have secured the services of Professor David Caldwell of the University of California, Santa Barbara, to write the new material in Chapters 17 and 18, as well as Appendix R. Only a person who has been totally immersed in research in particle physics could have done what had to be done to produce a brief but understandable treatment of what has happened in that field in recent years. Furthermore, since Caldwell is a colleague of the senior author, it was easy to have the interaction required to be sure that this new material was closely integrated into the earlier parts of the book, both in style and in content. Prepublication reviews have made it clear that Caldwell's material is a very strong addition to the book.

Professor Richard Christman, of the U.S. Coast Guard Academy, wrote the new material in Section 13-8, Section 16-6, and Appendix Q, receiving significant input from the authors. We are very pleased with the results.

The answers to selected problems, found in Appendix S, were prepared by Professor Edward Derringh, of the Wentworth Institute of Technology. He also edited the new additions to the problem sets and prepared a manual giving detailed solutions to most of the problems. The solutions manual is available to instructors from the publisher.

It is a pleasure to express our deep appreciation to the people mentioned above. We also thank Frank T. Avignone, III, University of South Carolina; Edward Cecil, Colorado School of Mines; L. Edward Millet, California State University, Chico; and James T. Tough, The Ohio State University, for their very useful prepublication reviews.

The following people offered suggestions or comments which helped in the development of the second edition: Alan H. Barrett, Massachusetts Institute of Technology; Richard H. Behrman, Swarthmore College; George F. Bertsch, Michigan State University; Richard N. Boyd, The Ohio State University; Philip A. Casabella, Rensselaer Polytechnic Institute; C. Dewey Cooper, University of Georgia; James E. Draper, University of California at Davis; Arnold Engler, Carnegie-Mellon University; A. T. Fromhold, Jr., Auburn University; Ross Garrett, University of Auckland; Russell Hobbie, University of Minnesota; Bei-Lok Hu, University of Maryland; Hillard Huntington, Rensselaer Polytechnic Institute; Mario Iona, University of Denver; Ronald G. Johnson, Trent University; A. L. Laskar, Clemson University; Charles W. Leming, Henderson State University; Luc Leplae, University of Wisconsin-Milwaukee; Ralph D. Meeker, Illinois Benedictine College; Roger N. Metz, Colby College; Ichiro Miyagawa, University of Alabama; J. A. Moore, Brock University; John J. O'Dwyer, State University of New York at Oswego; Douglas M. Potter, Rutgers State University; Russell A. Schaffer, Lehigh University; John W. Watson, Kent State University; and Robert White, University of Auckland. We appreciate their contribution.

Santa Barbara, California **Robert Eisberg**
Troy, New York **Robert Resnick**

PREFACE TO THE FIRST EDITION

The basic purpose of this book is to present clear and valid treatments of the properties of almost all of the important quantum systems from the point of view of elementary quantum mechanics. Only as much quantum mechanics is developed as is required to accomplish the purpose. Thus we have chosen to emphasize the applications of the theory more than the theory itself. In so doing we hope that the book will be well adapted to the attitudes of contemporary students in a terminal course on the phenomena of quantum physics. As students obtain an insight into the tremendous explanatory power of quantum mechanics, they should be motivated to learn more about the theory. Hence we hope that the book will be equally well adapted to a course that is to be followed by a more advanced course in formal quantum mechanics.

The book is intended primarily to be used in a one year course for students who have been through substantial treatments of elementary differential and integral calculus and of calculus level elementary classical physics. But it can also be used in shorter courses. Chapters 1 through 4 introduce the various phenomena of early quantum physics and develop the essential ideas of the old quantum theory. These chapters can be gone through fairly rapidly, particularly for students who have had some prior exposure to quantum physics. The basic core of quantum mechanics, and its application to one- and two-electron atoms, is contained in Chapters 5 through 8 and the first four sections of Chapter 9. This core can be covered well in appreciably less than half a year. Thus the instructor can construct a variety of shorter courses by adding to the core material from the chapters covering the essentially independent topics: multielectron atoms and molecules, quantum statistics and solids, nuclei and particles.

Instructors who require a similar but more extensive and higher level treatment of quantum mechanics, and who can accept a much more restricted coverage of the applications of the theory, may want to use *Fundamentals of Modern Physics* by Robert Eisberg (John Wiley & Sons, 1961), instead of this book. For instructors requiring a more comprehensive treatment of special relativity than is given in Appendix A, but similar in level and pedagogic style to this book, we recommend using in addition *Introduction to Special Relativity* by Robert Resnick (John Wiley & Sons, 1968).

Successive preliminary editions of this book were developed by us through a procedure involving intensive classroom testing in our home institutions and four other schools. Robert Eisberg then completed the writing by significantly revising and extending the last preliminary edition. He is consequently the senior author of this book. Robert Resnick has taken the lead in developing and revising the last preliminary edition so as to prepare the manuscript for a modern physics counterpart at a somewhat lower level. He will consequently be that book's senior author.

The pedagogic features of the book, some of which are not usually found in books at this level, were proven in the classroom testing to be very successful. These features are: detailed outlines at the beginning of each chapter, numerous worked out examples in each chapter, optional sections in the chapters and optional appendices, summary sections and tables, sets of questions at the end of each chapter, and long and varied sets of thoroughly tested problems at the end of each chapter, with subsets of answers at the end of the book. The writing is careful and expansive. Hence we believe that the book is well suited to self-learning and to self-paced courses.

We have employed the MKS (or SI) system of units, but not slavishly so. Where general practice in a particular field involves the use of alternative units, they are used here.

It is a pleasure to express our appreciation to Drs. Harriet Forster, Russell Hobbie, Stuart Meyer, Gerhard Salinger, and Paul Yergin for constructive reviews, to Dr. David Swedlow for assistance with the evaluation and solutions of the problems, to Dr. Benjamin Chi for assistance with the figures, to Mr. Donald Deneck for editorial and other assistance, and to Mrs. Cassie Young and Mrs. Carolyn Clemente for typing and other secretarial services.

Santa Barbara, California
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Robert Resnick

CONTENTS

1

THERMAL RADIATION AND PLANCK'S POSTULATE

1-1 INTRODUCTION

At a meeting of the German Physical Society on Dec. 14, 1900, Max Planck read his paper, "On the Theory of the Energy Distribution Law of the Normal Spectrum." This paper, which first attracted little attention, was the start of a revolution in physics. The date of its presentation is considered to be the birthday of quantum physics, although it was not until a quarter of a century later that modern quantum mechanics, the basis of our present understanding, was developed by Schroedinger and others. Many paths converged on this understanding, each showing another aspect of the breakdown of classical physics. In this and the following three chapters we shall examine the major milestones, of what is now called the *old quantum theory*, that led to modern quantum mechanics. The experimental phenomena which we shall discuss in connection with the old quantum theory span all the disciplines of classical physics: mechanics, thermodynamics, statistical mechanics, and electromagnetism. Their repeated contradiction of classical laws, and the resolution of these conflicts on the basis of quantum ideas, will show us the need for quantum mechanics. And our study of the old quantum theory will allow us to more easily obtain a deeper understanding of quantum mechanics when we begin to consider it in the fifth chapter.

As is true of relativity (which is treated briefly in Appendix A), quantum physics represents a generalization of classical physics that includes the classical laws as special cases. Just as relativity extends the range of application of physical laws to the region of high velocities, so quantum physics extends that range to the region of small dimensions. And just as a universal constant of fundamental significance, the velocity of light c, characterizes relativity, so a universal constant of fundamental significance, now called Planck's constant h, characterizes quantum physics. It was while trying to explain the observed properties of thermal radiation that Planck introduced this constant in his 1900 paper. Let us now begin to examine thermal radiation ourselves. We shall be led thereby to Planck's constant and the extremely significant related quantum concept of the discreteness of energy. We shall also find that thermal radiation has considerable importance and contemporary relevance in its own right. For instance, the phenomenon has recently helped astrophysicists decide among competing theories of the origin of the universe. Another example is given by the rapidly developing technology of solar heating, which depends on the thermal radiation received by the earth from the sun.

1-2 THERMAL RADIATION

The radiation emitted by a body as a result of its temperature is called *thermal radiation*. All bodies emit such radiation to their surroundings and absorb such radiation from them. If a body is at first hotter than its surroundings, it will cool off because its rate of emitting energy exceeds its rate of absorbing energy. When thermal equilibrium is reached the rates of emission and absorption are equal.

Matter in a condensed state (i.e., solid or liquid) emits a continuous spectrum of radiation. The details of the spectrum are almost independent of the particular material of which a body is composed, but they depend strongly on the temperature. At ordinary temperatures most bodies are visible to us not by their emitted light but by the light they reflect. If no light shines on them we cannot see them. At very high temperatures, however, bodies are self-luminous. We can see them glow in a darkened room; but even at temperatures as high as several thousand degrees Kelvin well over 90% of the emitted thermal radiation is invisible to us, being in the infrared part of the electromagnetic spectrum. Therefore, self-luminous bodies are quite hot.

Consider, for example, heating an iron poker to higher and higher temperatures in a fire, periodically withdrawing the poker from the fire long enough to observe its properties. When the poker is still at a relatively low temperature it radiates heat, but it is not visibly hot. With increasing temperature the amount of radiation that the poker emits increases very rapidly and visible effects are noted. The poker assumes a dull red color, then a bright red color, and, at very high temperatures, an intense blue-white color. That is, with increasing temperature the body emits more thermal radiation and the frequency of the most intense radiation becomes higher.

The relation between the temperature of a body and the frequency spectrum of the emitted radiation is used in a device called an optical pyrometer. This is essentially a rudimentary spectrometer that allows the operator to estimate the temperature of a hot body, such as a star, by observing the color, or frequency composition, of the thermal radiation that it emits. There is a continuous spectrum of radiation emitted, the eye seeing chiefly the color corresponding to the most intense emission in the visible region. Familiar examples of objects which emit visible radiation include hot coals, lamp filaments, and the sun.

Generally speaking, the detailed form of the spectrum of the thermal radiation emitted by a hot body depends somewhat upon the composition of the body. However, experiment shows that there is one class of hot bodies that emits thermal spectra of a universal character. These are called *blackbodies*, that is, bodies that have surfaces which absorb all the thermal radiation incident upon them. The name is appropriate because such bodies do not reflect light and appear black when their temperatures are low enough that they are not self-luminous. One example of a (nearly) blackbody would be *any* object coated with a diffuse layer of black pigment, such as lamp black or bismuth black. Another, quite different, example will be described shortly. Independent of the details of their composition, it is found that *all* blackbodies at the same temperature emit thermal radiation with the same spectrum. This general fact can be understood on the basis of classical arguments involving thermodynamic equilibrium. The specific form of the spectrum, however, cannot be obtained from thermodynamic arguments alone. The universal properties of the radiation emitted by blackbodies make them of particular theoretical interest and physicists sought to explain the specific features of their spectrum.

The spectral distribution of blackbody radiation is specified by the quantity $R_T(\nu)$, called the *spectral radiancy*, which is defined so that $R_T(\nu)\,d\nu$ is equal to the energy emitted per unit time in radiation of frequency in the interval ν to $\nu + d\nu$ from a unit area of the surface at absolute temperature T. The earliest accurate measurements of this quantity were made by Lummer and Pringsheim in 1899. They used an instrument essentially similar to the prism spectrometers used in measuring optical spectra, except that special materials were required for the lenses, prisms, etc., so that they would be transparent to the relatively low frequency thermal radiation. The experimentally observed dependence of $R_T(\nu)$ on ν and T is shown in Figure 1-1.

Distribution functions, of which spectral radiancy is an example, are very common in physics. For example, the Maxwellian speed distribution function (which looks rather like one of the curves in Figure 1-1) tells us how the molecules in a gas at a fixed pressure and temperature are distributed according to their speed. Another distribution function that the student has probably already seen is the one (which has the form of a decreasing exponential) specifying the times of decay of radioactive nuclei in a sample containing nuclei of a given species, and he has certainly seen a distribution function for the grades received on a physics exam.

The spectral radiancy distribution function of Figure 1-1 for a blackbody of a given area and a particular temperature, say 1000°K, shows us that: (1) there is very little power radiated in a frequency interval of fixed size $d\nu$ if that interval is at a frequency ν which is very small compared to 10^{14} Hz. The power is zero for ν equal to zero. (2) The power radiated in the interval $d\nu$ increases rapidly as ν increases from very small values. (3) It maximizes for a value of $\nu \simeq 1.1 \times 10^{14}$ Hz. That is, the radiated power is most intense at that frequency. (4) Above $\simeq 1.1 \times 10^{14}$ Hz the radiated power drops slowly but continuously as ν increases. It is zero again when ν approaches infinitely large values.

The two distribution functions for the higher values of temperature, 1500°K and 2000°K, displayed in the figure show us that (5) the frequency at which the radiated power is most

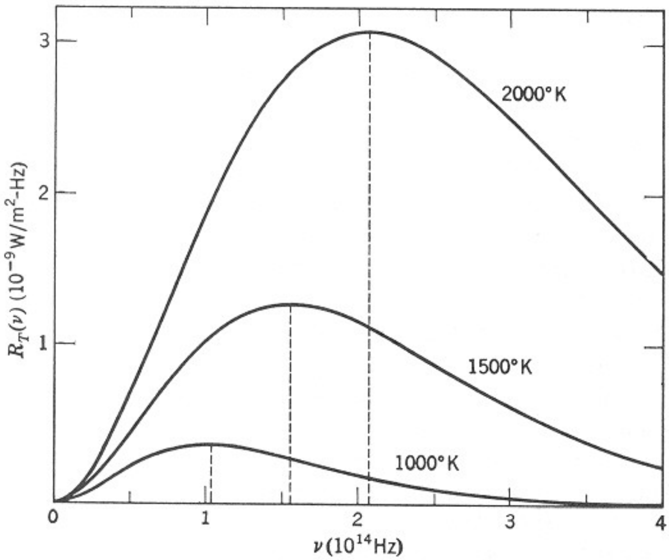

Figure 1-1 The spectral radiancy of a blackbody radiator as a function of the frequency of radiation, shown for temperatures of the radiator of 1000°K, 1500°K, and 2000°K. Note that the frequency at which the maximum radiancy occurs (dashed line) increases linearly with increasing temperature, and that the total power emitted per square meter of the radiator (area under curve) increases very rapidly with temperature.

intense increases with increasing temperature. Inspection will verify that this frequency increases linearly with temperature. (6) The total power radiated in all frequencies increases with increasing temperature, and it does so more rapidly than linearly. The total power radiated at a particular temperature is given simply by the area under the curve for that temperature, $\int_0^\infty R_T(\nu)\,d\nu$, since $R_T(\nu)\,d\nu$ is the power radiated in the frequency interval from ν to $\nu + d\nu$.

The integral of the spectral radiancy $R_T(\nu)$ over all ν is the total energy emitted per unit time per unit area from a blackbody at temperature T. It is called the *radiancy* R_T. That is

$$R_T = \int_0^\infty R_T(\nu)\,d\nu \tag{1-1}$$

As we have seen in the preceding discussion of Figure 1-1, R_T increases rapidly with increasing temperature. In fact, this result is called *Stefan's law*, and it was first stated in 1879 in the form of an empirical equation

$$R_T = \sigma T^4 \tag{1-2}$$

where

$$\sigma = 5.67 \times 10^{-8}\ \text{W/m}^2\text{-}^\circ\text{K}^4$$

is called the *Stefan-Boltzmann constant*. Figure 1-1 also shows us that the spectrum shifts toward higher frequencies as T increases. This result is called *Wien's displacement law*

$$\nu_{max} \propto T \tag{1-3a}$$

where ν_{max} is the frequency ν at which $R_T(\nu)$ has its maximum value for a particular T. As T increases, ν_{max} is displaced toward higher frequencies. All these results are in agreement with the familiar experiences discussed earlier, namely that the amount of thermal radiation emitted increases rapidly (the poker radiates much more heat energy at higher temperatures), and the principal frequency of the radiation becomes higher (the poker changes color from dull red to blue-white), with increasing temperature.

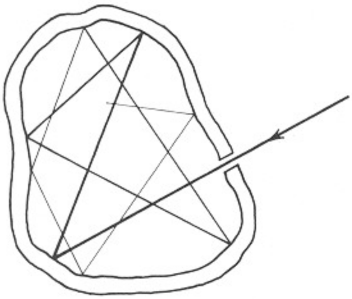

Figure 1-2 A cavity in a body connected by a small hole to the outside. Radiation incident on the hole is completely absorbed after successive reflections on the inner surface of the cavity. The hole absorbs like a blackbody. In the reverse process, in which radiation leaving the hole is built up of contributions emitted from the inner surface, the hole emits like a blackbody.

Another example of a blackbody, which we shall see to be particularly important, can be found by considering an object containing a cavity which is connected to the outside by a small hole, as in Figure 1-2. Radiation incident upon the hole from the outside enters the cavity and is reflected back and forth by the walls of the cavity, eventually being absorbed on these walls. If the area of the hole is very small compared to the area of the inner surface of the cavity, a negligible amount of the incident radiation will be reflected back through the hole. Essentially all the radiation incident upon the hole is absorbed; therefore, the *hole* must have the properties of the surface of a blackbody. Most blackbodies used in laboratory experiments are constructed along these lines.

Now assume that the walls of the cavity are uniformly heated to a temperature T. Then the walls will emit thermal radiation which will fill the cavity. The small fraction of this radiation incident from the inside upon the hole will pass through the hole. Thus the hole will act as an emitter of thermal radiation. Since the hole must have the properties of the surface of a blackbody, the radiation emitted by the hole must have a blackbody spectrum; but since the hole is merely sampling the thermal radiation present inside the cavity, it is clear that the radiation in the cavity must also have a blackbody spectrum. In fact, it will have a blackbody spectrum characteristic of the temperature T on the walls, since this is the only temperature defined for the system. The spectrum emitted by the hole in the cavity is specified in terms of the energy flux $R_T(\nu)$. It is more useful, however, to specify the spectrum of radiation inside the cavity, called *cavity radiation*, in terms of an *energy density*, $\rho_T(\nu)$, which is defined as the energy contained in a unit volume of the cavity at temperature T in the frequency interval ν to $\nu + d\nu$. It is evident that these quantities are proportional to one another; that is

$$\rho_T(\nu) \propto R_T(\nu) \tag{1-4}$$

Hence, the radiation inside a cavity whose walls are at temperature T has the same character as the radiation emitted by the surface of a blackbody at temperature T. It is convenient experimentally to produce a blackbody spectrum by means of a cavity in a heated body with a hole to the outside, and it is convenient in theoretical work to study blackbody radiation by analyzing the cavity radiation because it is possible to apply very general arguments to predict the properties of cavity radiation.

Example 1-1. (a) Since $\lambda\nu = c$, the constant velocity of light, Wien's displacement law (1-3a) can also be put in the form

$$\lambda_{\max} T = \text{const} \tag{1-3b}$$

where $\lambda_{\max}$ is the wavelength at which the spectral radiancy has its maximum value for a particular temperature T. The experimentally determined value of Wien's constant is 2.898×10^{-3} m-°K. If we assume that stellar surfaces behave like blackbodies we can get a good estimate of their temperature by measuring $\lambda_{\max}$. For the sun $\lambda_{\max} = 5100$ Å, whereas for the North Star $\lambda_{\max} = 3500$ Å. Find the surface temperature of these stars. (One *angstrom* = 1 Å = 10^{-10} m.)

►For the sun, $T = 2.898 \times 10^{-3}$ m-°K/5100×10^{-10} m $= 5700$°K. For the North Star, $T = 2.898 \times 10^{-3}$ m-°K/3500×10^{-10} m $= 8300$°K.

At 5700°K the sun's surface is near the temperature at which the greatest part of its radiation lies within the visible region of the spectrum. This suggests that over the ages of human evolution our eyes have adapted to the sun to become most sensitive to those wavelengths which it radiates most intensely. ◄

(b) Using Stefan's law, (1-2), and the temperatures just obtained, determine the power radiated from 1 cm^2 of stellar surface.

►For the sun

$$R_T = \sigma T^4 = 5.67 \times 10^{-8}\ \text{W/m}^2\text{-°K}^4 \times (5700\text{°K})^4$$
$$= 5.90 \times 10^7\ \text{W/m}^2 \simeq 6000\ \text{W/cm}^2$$

For the North Star

$$R_T = \sigma T^4 = 5.67 \times 10^{-8}\ \text{W/m}^2\text{-°K}^4 \times (8300\text{°K})^4$$
$$= 2.71 \times 10^8\ \text{W/m}^2 \simeq 27{,}000\ \text{W/cm}^2$$ ◄

Example 1-2. Assume we have two small opaque bodies a large distance from one another supported by fine threads in a large evacuated enclosure whose walls are opaque and kept at a constant temperature. In such a case the bodies and walls can exchange energy only by means of radiation. Let e represent the rate of emission of radiant energy by a body and let a represent the rate of absorption of radiant energy by a body. Show that at equilibrium

$$\frac{e_1}{a_1} = \frac{e_2}{a_2} = 1 \tag{1-5}$$

This relation, (1-5), is known as *Kirchhoff's law for radiation.*

►The equilibrium state is one of constant temperature throughout the enclosed system, and in that state the emission rate necessarily equals the absorption rate for each body. Hence

$$e_1 = a_1 \qquad \text{and} \qquad e_2 = a_2$$

Therefore

$$\frac{e_1}{a_1} = 1 = \frac{e_2}{a_2}$$

If one body, say body 2, is a blackbody, then $a_2 > a_1$ because a blackbody is a better absorber than a non-blackbody. Hence, it follows from (1-5) that $e_2 > e_1$. The observed fact that good absorbers are also good emitters is thus predicted by Kirchhoff's law. ◄

1-3 CLASSICAL THEORY OF CAVITY RADIATION

Shortly after the turn of the present century, Rayleigh, and also Jeans, made a calculation of the energy density of cavity (or blackbody) radiation that points up a serious conflict between classical physics and experimental results. This calculation is similar to calculations that arise in considering many other phenomena (e.g., specific heats of solids) to be treated later. We present the details here, but as an aid in guiding us through the calculations we first outline their general procedure.

Consider a cavity with metallic walls heated uniformly to temperature T. The walls emit electromagnetic radiation in the thermal range of frequencies. We know that this happens, basically, because of the accelerated motions of the electrons in the metallic walls that arise from thermal agitation (see Appendix B). However, it is not necessary to study the behavior of the electrons in the walls of the cavity in detail. Instead, attention is focused on the behavior of the electromagnetic waves in the interior of the cavity. Rayleigh and Jeans proceeded as follows. First, classical electromagnetic theory is used to show that the radiation inside the cavity must exist in the form of standing waves with nodes at the metallic surfaces. By using geometrical arguments, a count is made of the number of such standing waves in the frequency interval ν to $\nu + d\nu$, in order to determine how the number depends on ν. Then a result of classical kinetic theory is used to calculate the average total energy of these waves when the system is in thermal equilibrium. The average total energy depends, in the classical theory, only on the temperature T. The number of standing waves in the frequency interval times the average energy of the waves, divided by the volume of the cavity, gives the average energy content per unit volume in the frequency interval ν to $\nu + d\nu$. This is the required quantity, the energy density $\rho_T(\nu)$. Let us now do it ourselves.

We assume for simplicity that the metallic-walled cavity filled with electromagnetic radiation is in the form of a cube of edge length a, as shown in Figure 1-3. Then the radiation reflecting back and forth between the walls can be analyzed into three components along the three mutually perpendicular directions defined by the edges of the cavity. Since the opposing walls are parallel to each other, the three components of the radiation do not mix, and we may treat them separately. Consider first the x component and the metallic wall at $x = 0$. All the radiation of this component which is incident upon the wall is reflected by it, and the incident and reflected waves combine to form a standing wave. Now, since electromagnetic radiation is a transverse vibration with the electric field vector $\mathbf{E}$ perpendicular to the propagation direction, and since the propagation direction for this component is perpendicular to the wall in question, its electric field vector $\mathbf{E}$ is parallel to the wall. A metallic wall cannot, however, support an electric field parallel to the surface, since charges can always flow in such a way as to neutralize the electric field. Therefore, $\mathbf{E}$ for this component must always be zero at the wall. That is, the standing wave associated with the x-component of the radiation must have a node (zero amplitude) at $x = 0$. The standing wave must also have a node at $x = a$ because there can be no parallel electric field in the corresponding wall. Furthermore, similar conditions apply to the other two components; the standing wave associated with the y component must have nodes at $y = 0$ and $y = a$, and the standing wave associated with the z component must have nodes at $z = 0$ and $z = a$. These conditions put a limitation on the possible wavelengths, and therefore on the possible frequencies, of the electromagnetic radiation in the cavity.

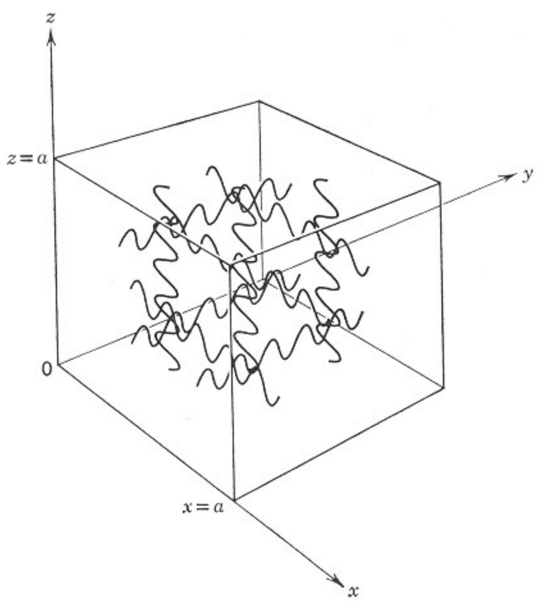

Figure 1-3 A metallic walled cubical cavity filled with electromagnetic radiation, showing three noninterfering components of that radiation bouncing back and forth between the walls and forming standing waves with nodes at each wall.

Now we shall consider the question of counting the number of standing waves with nodes on the surfaces of the cavity, whose wavelengths lie in the interval λ to $\lambda + d\lambda$ corresponding to the frequency interval ν to $\nu + d\nu$. To focus attention on the ideas involved in the calculation, we shall first treat the x component alone; that is, we shall consider the simplified, but artificial, case of a "one-dimensional cavity" of length a. After we have worked through this case, we shall see that the procedure for generalizing to a real three-dimensional cavity is obvious.

The electric field for one-dimensional electromagnetic standing waves can be described mathematically by the function

$$E(x,t) = E_0 \sin(2\pi x/\lambda) \sin(2\pi \nu t) \tag{1-6}$$

where λ is the wavelength of the wave, ν is its frequency, and E_0 is its maximum amplitude. The first two quantities are related by the equation

$$\nu = c/\lambda \tag{1-7}$$

where c is the propagation velocity of electromagnetic waves. Equation (1-6) represents a wave whose amplitude has the sinusoidal space variation $\sin(2\pi x/\lambda)$ and which is oscillating in time sinusoidally with frequency ν like a simple harmonic oscillator. Since the amplitude is obviously zero, at all times t, for positions satisfying the relation

$$2x/\lambda = 0, 1, 2, 3, \ldots \tag{1-8}$$

the wave has fixed nodes; that is, it is a standing wave. In order to satisfy the requirement that the waves have nodes at both ends of the one-dimensional cavity, we choose the origin of the x axis to be at one end of the cavity ($x = 0$) and then require that at the other end ($x = a$)

$$2x/\lambda = n \qquad \text{for } x = a \tag{1-9}$$

where

$$n = 1, 2, 3, 4, \ldots$$

This condition determines a set of allowed values of the wavelength λ. For these allowed values, the amplitude patterns of the standing waves have the appearance shown in Figure 1-4. These patterns may be recognized as the standing wave patterns for vibrations of a string fixed at both ends, a real physical system which also satisfies (1-6). In our case the patterns represent electromagnetic standing waves.

It is convenient to continue the discussion in terms of the allowed frequencies instead of the allowed wavelengths. These frequencies are $\nu = c/\lambda$, where $2a/\lambda = n$. That is

$$\nu = cn/2a \qquad n = 1, 2, 3, 4, \ldots \tag{1-10}$$

We can represent these allowed values of frequency in terms of a diagram consisting of an axis on which we plot a point at every integral value of n. On such a diagram, the value of the allowed frequency ν corresponding to a particular value of n is, by (1-10), equal to $c/2a$ times the distance d from the origin to the appropriate point, or the distance d is $2a/c$ times the frequency ν. These relations are shown in Figure 1-5. Such a diagram is useful in calculating the number of allowed values in frequency

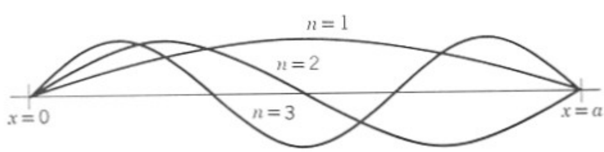

Figure 1-4 The amplitude patterns of standing waves in a one-dimensional cavity with walls at $x = 0$ and $x = a$, for the first three values of the index n.

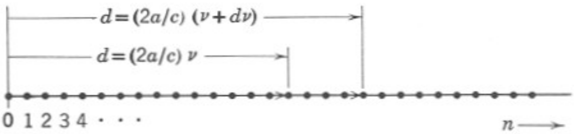

Figure 1-5 The allowed values of the index n, which determines the allowed values of the frequency, in a one-dimensional cavity of length a.

range ν to $\nu + d\nu$, which we call $N(\nu)\,d\nu$. To evaluate this quantity we simply count the number of points on the n axis which fall between two limits which are constructed so as to correspond to the frequencies ν and $\nu + d\nu$, respectively. Since the points are distributed uniformly along the n axis, it is apparent that the number of points falling between the two limits will be proportional to $d\nu$ but will not depend on ν. In fact, it is easy to see that $N(\nu)\,d\nu = (2a/c)\,d\nu$. However, we must multiply this by an additional factor of 2 since, for each of the allowed frequencies, there are actually two independent waves corresponding to the two possible states of polarization of electromagnetic waves. Thus we have

$$N(\nu)\,d\nu = \frac{4a}{c}\,d\nu \tag{1-11}$$

This completes the calculation of the number of allowed standing waves for the artificial case of a one-dimensional cavity.

The above calculation makes apparent the procedures for extending the calculation to the real case of a three-dimensional cavity. This extension is indicated in Figure 1-6. Here the set of points uniformly distributed at integral values along a single n axis is replaced by a uniform three-dimensional array of points whose three coordinates occur at integral values along each of three mutually perpendicular n axes. *Each point* of the array corresponds to a particular allowed three-dimensional

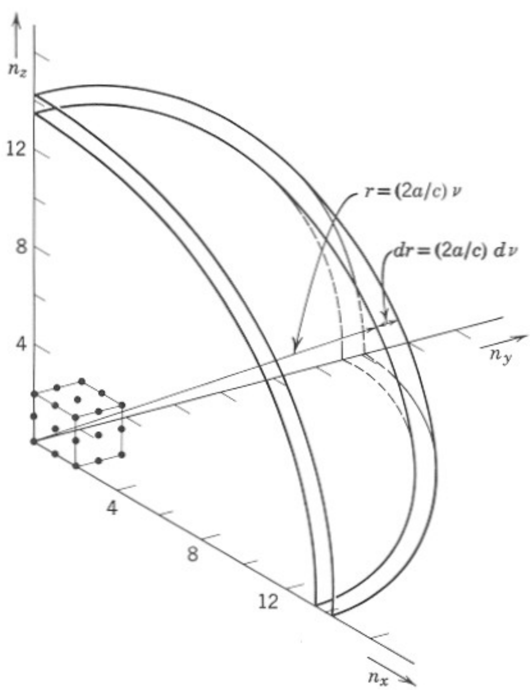

Figure 1-6 The allowed frequencies in a three-dimensional cavity in the form of a cube of edge length a are determined by three indices n_x, n_y, n_z, which can each assume only integral values. For clarity, only a few of the very many points corresponding to sets of these indices are shown.

standing wave. The integral values of n_x, n_y, and n_z specified by each point give the number of nodes of the x, y, and z components, respectively, of the three-dimensional wave. The procedure is equivalent to analyzing a three-dimensional wave (i.e., one propagated in an arbitrary direction) into three one-dimensional component waves. Here the number of allowed frequencies in the frequency interval ν to $\nu + d\nu$ is equal to the number of points contained between shells of radii corresponding to frequencies ν and $\nu + d\nu$, respectively. This will be proportional to the volume contained between these two shells, since the points are uniformly distributed. Thus it is apparent that $N(\nu)\,d\nu$ will be proportional to $\nu^2\,d\nu$, the first factor, ν^2, being proportional to the area of the shells and the second factor, $d\nu$, being the distance between them. In the following example we shall work out the details and find

$$N(\nu)\,d\nu = \frac{8\pi V}{c^3}\,\nu^2\,d\nu \tag{1-12}$$

where $V = a^3$, the volume of the cavity.

Example 1-3. Derive (1-12), which gives the number of allowed electromagnetic standing waves in each frequency interval for the case of a three-dimensional cavity in the form of a metallic-walled cube of edge length a.

▶Consider radiation of wavelength λ and frequency $\nu = c/\lambda$, propagating in the direction defined by the three angles α, β, γ, as shown in Figure 1-7. The radiation must be a standing wave since all three of its components are standing waves. We have indicated the locations of some of the fixed nodes of this standing wave by a set of planes perpendicular to the propagation direction α, β, γ. The distance between these nodal planes of the radiation is just $\lambda/2$, where λ is its wavelength. We have also indicated the locations at the three axes of the nodes of the three components. The distances between these nodes are

$$\begin{aligned} \lambda_x/2 &= \lambda/2\cos\alpha \\ \lambda_y/2 &= \lambda/2\cos\beta \\ \lambda_z/2 &= \lambda/2\cos\gamma \end{aligned} \tag{1-13}$$

Let us write expressions for the magnitudes at the three axes of the electric fields of the three components. They are

$$\begin{aligned} E(x,t) &= E_{0_x}\sin(2\pi x/\lambda_x)\sin(2\pi\nu t) \\ E(y,t) &= E_{0_y}\sin(2\pi y/\lambda_y)\sin(2\pi\nu t) \\ E(z,t) &= E_{0_z}\sin(2\pi z/\lambda_z)\sin(2\pi\nu t) \end{aligned}$$

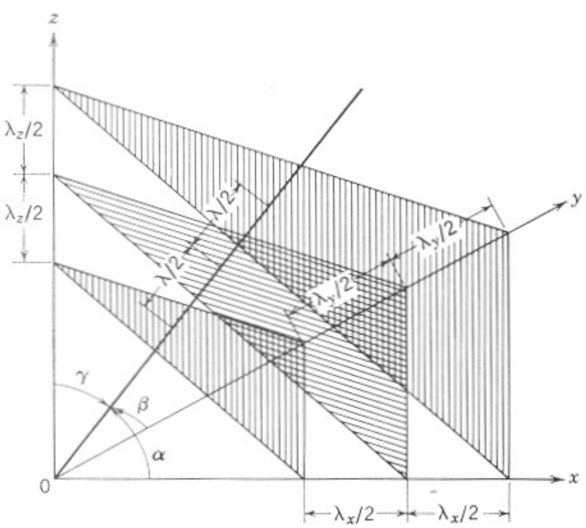

Figure 1-7 The nodal planes of a standing wave propagating in a certain direction in a cubical cavity.

The expression for the x component represents a wave with a maximum amplitude E_{0_x}, with a space variation $\sin(2\pi x/\lambda_x)$, and which is oscillating with frequency ν. As $\sin(2\pi x/\lambda_x)$ is zero for $2x/\lambda_x = 0, 1, 2, 3, \ldots$, the wave is a standing wave of wavelength λ_x because it has fixed nodes separated by the distance $\Delta x = \lambda_x/2$. The expressions for the y and z components represent standing waves of maximum amplitudes E_{0_y} and E_{0_z} and wavelengths λ_y and λ_z, but all three component standing waves oscillate with the frequency ν of the radiation. Note that these expressions automatically satisfy the requirement that the x component have a node at $x = 0$, the y component have a node at $y = 0$, and the z component have a node at $z = 0$. To make them also satisfy the requirement that the x component have a node at $x = a$, the y component have a node at $y = a$, and the z component have a node at $z = a$, set

$$2x/\lambda_x = n_x \qquad \text{for } x = a$$
$$2y/\lambda_y = n_y \qquad \text{for } y = a$$
$$2z/\lambda_z = n_z \qquad \text{for } z = a$$

where $n_x = 1, 2, 3, \ldots; n_y = 1, 2, 3, \ldots; n_z = 1, 2, 3, \ldots$. Using (1-13), these conditions become

$$(2a/\lambda)\cos\alpha = n_x \qquad (2a/\lambda)\cos\beta = n_y \qquad (2a/\lambda)\cos\gamma = n_z$$

Squaring both sides of these equations and adding, we obtain

$$(2a/\lambda)^2(\cos^2\alpha + \cos^2\beta + \cos^2\gamma) = n_x^2 + n_y^2 + n_z^2$$

but the angles α, β, γ have the property

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

Thus

$$2a/\lambda = \sqrt{n_x^2 + n_y^2 + n_z^2}$$

where n_x, n_y, n_z take on all possible integral values. This equation describes the limitation on the possible wavelengths of the electromagnetic radiation contained in the cavity.

We again continue the discussion in terms of the allowed frequencies instead of the allowed wavelengths. They are

$$\nu = \frac{c}{\lambda} = \frac{c}{2a}\sqrt{n_x^2 + n_y^2 + n_z^2} \tag{1-14a}$$

Now we shall count the number of allowed frequencies in a given frequency interval by constructing a uniform cubic lattice in one octant of a rectangular coordinate system in such a way that the three coordinates of each point of the lattice are equal to a possible set of the three integers n_x, n_y, n_z (see Figure 1-6). By construction, each lattice point corresponds to an allowed frequency. Furthermore, $N(\nu)d\nu$, the number of allowed frequencies between ν and $\nu + d\nu$, is equal to $N(r)\,dr$, the number of points contained between concentric shells of radii r and $r + dr$, where

$$r = \sqrt{n_x^2 + n_y^2 + n_z^2}$$

From (1-14a), this is

$$r = \frac{2a}{c}\nu \tag{1-14b}$$

Since $N(r)\,dr$ is equal to the volume enclosed by the shells times the density of lattice points, and since, by construction, the density is one, $N(r)\,dr$ is simply

$$N(r)\,dr = \frac{1}{8}4\pi r^2\,dr = \frac{\pi r^2\,dr}{2} \tag{1-15}$$

Setting this equal to $N(\nu)d\nu$, and evaluating $r^2\,dr$ from (1-14b), we have

$$N(\nu)\,d\nu = \frac{\pi}{2}\left(\frac{2a}{c}\right)^3 \nu^2\,d\nu$$

This completes the calculation except that we must multiply these results by a factor of 2 because, for each of the allowed frequencies we have enumerated, there are actually two independent waves corresponding to the two possible states of polarization of electromagnetic radiation. Thus we have derived (1-12). It can be shown that $N(\nu)$ is independent of the assumed shape of the cavity and depends only on its volume. ◀

Note that there is a very significant difference between the results obtained for the case of a real three-dimensional cavity and the results we obtained earlier for the artificial case of a one-dimensional cavity. The factor of ν^2 found in (1-12), but not in (1-11), will be seen to play a fundamental role in the arguments that follow. This factor arises, basically, because we live in a three-dimensional world—the power of ν being one less than the dimensionality. Although Planck, in ultimately resolving the serious discrepancies between classical theory and experiment, had to question certain points which had been considered to be obviously true, neither he nor others working on the problem questioned (1-12). It was, and remains, generally agreed that (1-12) is valid.

We now have a count of the number of standing waves. The next step in the Rayleigh-Jeans classical theory of blackbody radiation is the evaluation of the average total energy contained in each standing wave of frequency ν. According to classical physics, the energy of some particular wave can have any value from zero to infinity, the actual value being proportional to the square of the magnitude of its amplitude constant E_0. However, for a system containing a large number of physical entities of the same kind which are in thermal equilibrium with each other at temperature T, classical physics makes a very definite prediction about the *average* values of the energies of the entities. This applies to our case since the multitude of standing waves, which constitute the thermal radiation inside the cavity, are entities of the same kind which are in thermal equilibrium with each other at the temperature T of the walls of the cavity. Thermal equilibrium is ensured by the fact that the walls of any real cavity will always absorb and reradiate, in different frequencies and directions, a small amount of the radiation incident upon them and, therefore, the different standing waves can gradually exchange energy as required to maintain equilibrium.

The prediction comes from classical kinetic theory, and it is called the *law of equipartition of energy*. This law states that for a system of gas molecules in thermal equilibrium at temperature T, the average kinetic energy of a molecule per degree of freedom is $kT/2$, where $k = 1.38 \times 10^{-23}$ joule/°K is called *Boltzmann's constant*. The law actually applies to any classical system containing, in equilibrium, a large number of entities of the same kind. For the case at hand the entities are standing waves which have one degree of freedom, their electric field amplitudes. Therefore, on the average their *kinetic* energies all have the same value, $kT/2$. However, each sinusoidally oscillating standing wave has a *total* energy which is twice its average kinetic energy. This is a common property of physical systems which have a single degree of freedom that execute simple harmonic oscillations in time; familiar cases are a pendulum or a coil spring. Thus each standing wave in the cavity has, according to the classical equipartition law, an average total energy

$$\overline{\mathscr{E}} = kT \tag{1-16}$$

The most important point to note is that the average total energy $\overline{\mathscr{E}}$ is predicted to have the same value for all standing waves in the cavity, independent of their frequencies.

The energy per unit volume in the frequency interval ν to $\nu + d\nu$ of the blackbody spectrum of a cavity at temperature T is just the product of the average energy per standing wave times the number of standing waves in the frequency interval, divided by the volume of the cavity. From (1-15) and (1-16) we therefore finally obtain the result

$$\rho_T(\nu)\,d\nu = \frac{8\pi\nu^2 kT}{c^3}\,d\nu \tag{1-17}$$

This the *Rayleigh-Jeans formula for blackbody radiation.*

In Figure 1-8 we compare the predictions of this equation with experimental data. The discrepancy is apparent. In the limit of low frequencies, the classical spectrum approaches the experimental results, but, as the frequency becomes large, the theoretical prediction goes to infinity! Experiment shows that the energy density always

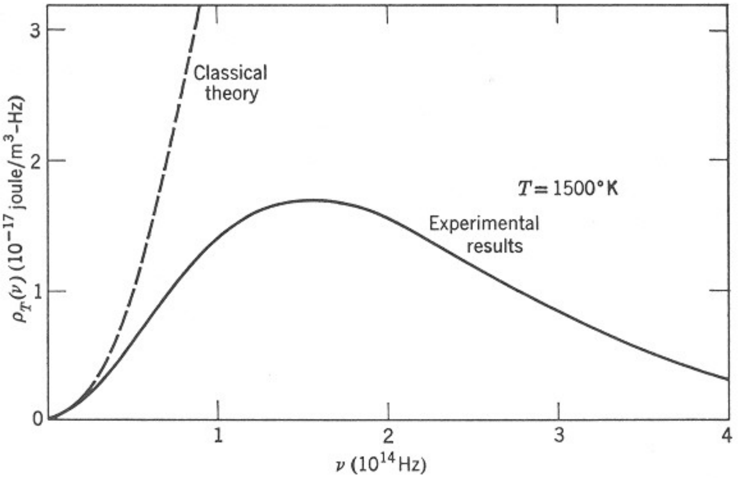

Figure 1-8 The Rayleigh-Jeans prediction (dashed line) compared with the experimental results (solid line) for the energy density of a blackbody cavity, showing the serious discrepancy called the ultraviolet catastrophe.

remains finite, as it obviously must, and, in fact, that the energy density goes to zero at very high frequencies. The grossly unrealistic behavior of the prediction of classical theory at high frequencies is known in physics as the "ultraviolet catastrophe." This term is suggestive of the importance of the failure of the theory.

1-4 PLANCK'S THEORY OF CAVITY RADIATION

In trying to resolve the discrepancy between theory and experiment, Planck was led to consider the possibility of a violation of the law of equipartition of energy on which the theory was based. From Figure 1-8 it is clear that the law gives satisfactory results for small frequencies. Thus we can assume

$$\bar{\mathscr{E}} \xrightarrow[\nu \to 0]{} kT \qquad \text{(1-18)}$$

that is, the average total energy approaches kT as the frequency approaches zero. The discrepancy at high frequencies could be eliminated if there is, for some reason, a cutoff, so that

$$\bar{\mathscr{E}} \xrightarrow[\nu \to \infty]{} 0 \qquad \text{(1-19)}$$

that is, if the average total energy approaches zero as the frequency approaches infinity. In other words, Planck realized that, in the circumstances that prevail for the case of blackbody radiation, the average energy of the standing waves is a function of frequency $\bar{\mathscr{E}}(\nu)$ having the properties indicated by (1-18) and (1-19). This is in contrast to the law of equipartition of energy which assigns to the average energy $\bar{\mathscr{E}}$ a value independent of frequency.

Let us look at the origin of the equipartition law. It arises, basically, from a more comprehensive result of classical statistical mechanics called the Boltzmann distribution. (Arguments leading to the Boltzmann distribution are given in Appendix C for students not already familiar with it.) Here we shall use a *special form of the Boltzmann distribution*

$$P(\mathscr{E}) = \frac{e^{-\mathscr{E}/kT}}{kT} \qquad \text{(1-20)}$$

in which $P(\mathscr{E})\,d\mathscr{E}$ is the probability of finding a given entity of a system with energy in the interval between $\mathscr{E}$ and $\mathscr{E} + d\mathscr{E}$, when the number of energy states for the entity in that interval is independent of $\mathscr{E}$. The system is supposed to contain a large number of entities of the same kind in thermal equilibrium at temperature T, and k represents Boltzmann's constant. The energies of the entities in the system we are considering, a set of simple harmonic oscillating standing waves in thermal equilibrium in a blackbody cavity, are governed by (1-20).

The Boltzmann distribution function is intimately related to Maxwell's distribution function for the energy of a molecule in a system of molecules in thermal equilibrium. In fact, the exponential in the Boltzmann distribution is responsible for the exponential factor in the Maxwell distribution. The factor of $\mathscr{E}^{1/2}$ that some students may know is also present in the Maxwell distribution results from the circumstance that the number of energy states for a molecule in the interval $\mathscr{E}$ to $\mathscr{E} + d\mathscr{E}$ is not independent of $\mathscr{E}$ but instead increases in proportion to $\mathscr{E}^{1/2}$.

The Boltzmann distribution function provides complete information about the energies of the entities in our system, including, of course, the average value $\bar{\mathscr{E}}$ of the energies. The latter quantity can be obtained from $P(\mathscr{E})$ by using (1-20) to evaluate the integrals in the ratio

$$\bar{\mathscr{E}} = \frac{\int_0^\infty \mathscr{E} P(\mathscr{E})\, d\mathscr{E}}{\int_0^\infty P(\mathscr{E})\, d\mathscr{E}} \tag{1-21}$$

The integrand in the numerator is the energy, $\mathscr{E}$, weighted by the probability that the entity will be found with this energy. By integrating over all possible energies, the average value of the energy is obtained. The denominator is the probability of finding the entity with *any* energy and so should have the value one; it does. The integral in the numerator can be evaluated, and the result is just the law of equipartition of energy

$$\bar{\mathscr{E}} = kT \tag{1-22}$$

Instead of actually carrying through the evaluation here, it will be better, for the purpose of arguments to follow, to look at the graphical presentation of $P(\mathscr{E})$ and $\bar{\mathscr{E}}$ shown in the top half of Figure 1-9. There $P(\mathscr{E})$ is plotted as a function of $\mathscr{E}$. Its maximum value, $1/kT$, occurs at $\mathscr{E} = 0$, and the value of $P(\mathscr{E})$ decreases smoothly with increasing $\mathscr{E}$ to approach zero as $\mathscr{E} \to \infty$. That is, the result that would most probably be found in a measurement of $\mathscr{E}$ is zero. But the average $\bar{\mathscr{E}}$ of the results that would be found in a number of measurements of $\mathscr{E}$ is greater than zero, as is shown on the abscissa of the top figure, since many measurements of $\mathscr{E}$ will lead to values greater than zero. The bottom half of Figure 1–9 indicates the evaluation of $\bar{\mathscr{E}}$ from $P(\mathscr{E})$.

Planck's great contribution came when he realized that he could obtain the required cutoff, indicated in (1-19), if he modified the calculation leading from $P(\mathscr{E})$ to $\bar{\mathscr{E}}$ by treating the energy $\mathscr{E}$ as if it were a *discrete variable* instead of as the *continuous variable* that it definitely is from the point of view of classical physics. Quantitatively, this can be done by rewriting (1-21) in terms of a sum instead of an integral. We shall soon see that this is not too hard to do, but it will be much more instructive for us to study the graphical presentation in Figure 1-10 first.

Planck assumed that the energy $\mathscr{E}$ could take on only certain discrete values, rather than any value, and that the discrete values of the energy were uniformly distributed; that is, he took

$$\mathscr{E} = 0, \Delta\mathscr{E}, 2\Delta\mathscr{E}, 3\Delta\mathscr{E}, 4\Delta\mathscr{E}, \ldots \tag{1-23}$$

as the set of allowed values of the energy. Here $\Delta\mathscr{E}$ is the uniform interval between

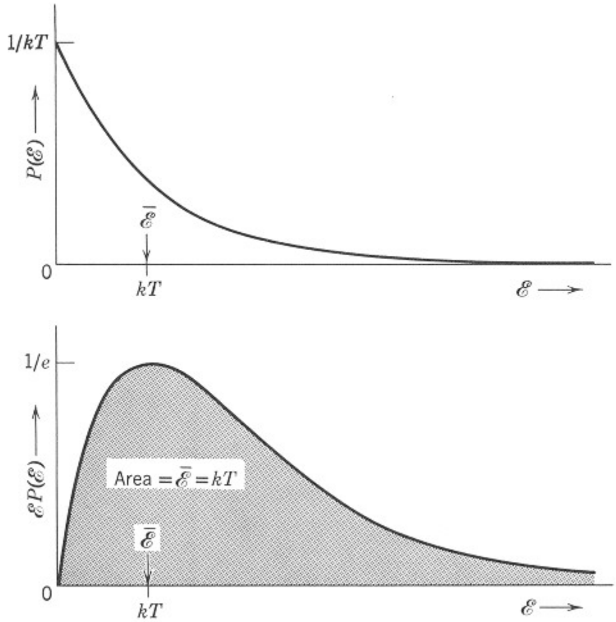

Figure 1-9 *Top:* A plot of the Boltzmann probability distribution $P(\mathscr{E}) = e^{-\mathscr{E}/kT}/kT$. The average value of the energy $\mathscr{E}$ for this distribution is $\overline{\mathscr{E}} = kT$, which is the classical law of equipartition of energy. To calculate this value of $\overline{\mathscr{E}}$, we integrate $\mathscr{E}P(\mathscr{E})$ from zero to infinity. This is just the quantity that is being averaged, $\mathscr{E}$, multiplied by the relative probability $P(\mathscr{E})$ that the value of $\mathscr{E}$ will be found in a measurement of the energy. *Bottom:* A plot of $\mathscr{E}P(\mathscr{E})$. The area under this curve gives the value of $\overline{\mathscr{E}}$.

successive allowed values of the energy. The top part of Figure 1-10 illustrates an evaluation of $\overline{\mathscr{E}}$ from $P(\mathscr{E})$, for a case in which $\Delta\mathscr{E} \ll kT$. In this case the result obtained is $\overline{\mathscr{E}} \simeq kT$. That is, a value essentially equal to the classical result is obtained here since the discreteness $\Delta\mathscr{E}$ is very small compared to the energy range kT in which $P(\mathscr{E})$ changes by a significant amount; it makes no essential difference in this case whether $\mathscr{E}$ is continuous or discrete. The middle part of Figure 1-10 illustrates the case in which $\Delta\mathscr{E} \simeq kT$. Here we find $\overline{\mathscr{E}} < kT$, because most of the entities have energy $\mathscr{E} = 0$ since $P(\mathscr{E})$ has a rather small value at the first allowed nonzero value $\Delta\mathscr{E}$ so $\mathscr{E} = 0$ dominates the calculation of the average value of $\mathscr{E}$ and a smaller result is obtained. The effect of the discreteness is seen most clearly, however, in the lower part of Figure 1-10, which illustrates a case in which $\Delta\mathscr{E} \gg kT$. In this case the probability of finding an entity with any of the allowed energy values greater than zero is negligible, since $P(\mathscr{E})$ is extremely small for all these values, and the result obtained is $\overline{\mathscr{E}} \ll kT$.

Recapitulating, Planck discovered that he could obtain $\overline{\mathscr{E}} \simeq kT$ when the difference in adjacent energies $\Delta\mathscr{E}$ is small, and $\overline{\mathscr{E}} \simeq 0$ when $\Delta\mathscr{E}$ is large. Since he needed to obtain the first result for small values of the frequency ν, and the second result for large values of ν, he clearly needed to make $\Delta\mathscr{E}$ an increasing function of ν. Numerical work showed him that he could take the simplest possible relation between $\Delta\mathscr{E}$ and ν having this property. That is, he assumed these quantities to be proportional

$$\Delta\mathscr{E} \propto \nu \tag{1-24}$$

Written as an equation instead of a proportionality, this is

$$\Delta\mathscr{E} = h\nu \tag{1-25}$$

where h is the proportionality constant.

Further numerical work allowed Planck to determine the value of the constant h by finding the value which produced the best fit of his theory with the experimental

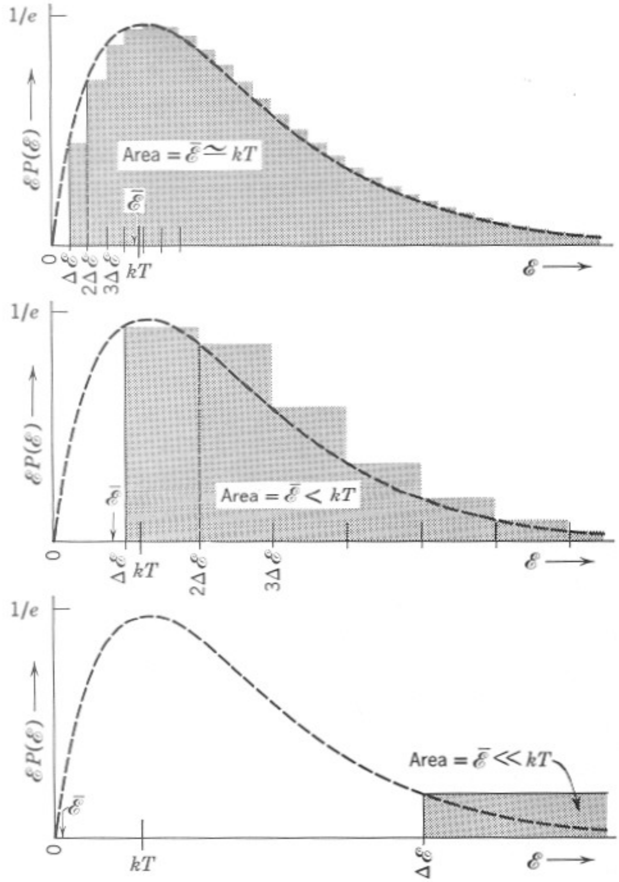

Figure 1-10 *Top:* If the energy $\mathscr{E}$ is not a continuous variable but is instead restricted to discrete values $0, \Delta\mathscr{E}, 2\Delta\mathscr{E}, 3\Delta\mathscr{E}, \ldots$, as indicated by the ticks on the $\mathscr{E}$ axis of the figure, the integral used to calculate the average value $\bar{\mathscr{E}}$ must be replaced by a summation. The average value is thus a sum of areas of rectangles, each of width $\Delta\mathscr{E}$, and with heights given by the allowed values of $\mathscr{E}$ times $P(\mathscr{E})$ at the beginning of each interval. In this figure $\Delta\mathscr{E} \ll kT$, and the allowed energies being closely spaced the area of all the rectangles differs but little from the area under the smooth curve. Thus the average value $\bar{\mathscr{E}}$ is nearly equal to kT, the value found in Figure 1-9. *Middle:* $\Delta\mathscr{E} \simeq kT$, and $\bar{\mathscr{E}}$ has a smaller value than it has in the case of the top figure. *Bottom:* $\Delta\mathscr{E} \gg kT$, and $\bar{\mathscr{E}}$ is further reduced. In all three figures the rectangles show the contribution to the total area of $\mathscr{E}P(\mathscr{E})$ for each allowed energy. The rectangle for $\mathscr{E} = 0$ of course is always of zero height. This will make a large effect on the total area if the widths of the rectangles are large.

data. The value he obtained was very close to the currently accepted value

$$h = 6.63 \times 10^{-34} \text{ joule-sec}$$

This very famous constant is now called *Planck's constant*.

The formula Planck obtained for $\bar{\mathscr{E}}$ by evaluating the summation analogous to the integral in (1-21), and that we shall obtain in Example 1-4, is

$$\bar{\mathscr{E}}(\nu) = \frac{h\nu}{e^{h\nu/kT} - 1} \tag{1-26}$$

Since $e^{h\nu/kT} \to 1 + h\nu/kT$ for $h\nu/kT \to 0$, we see that $\bar{\mathscr{E}}(\nu) \to kT$ in this limit as predicted by (1-18). In the limit $h\nu/kT \to \infty$, $e^{h\nu/kT} \to \infty$, and $\bar{\mathscr{E}}(\nu) \to 0$, in agreement with the prediction of (1-19).

The formula which he then immediately obtained for the energy density in the blackbody spectrum, using his result for $\bar{\mathscr{E}}(\nu)$ rather than the classical value $\bar{\mathscr{E}} = kT$, is

$$\rho_T(\nu)\,d\nu = \frac{8\pi\nu^2}{c^3}\frac{h\nu}{e^{h\nu/kT}-1}\,d\nu \tag{1-27}$$

This is *Planck's blackbody spectrum.* Figure 1-11 shows a comparison of this result of Planck's theory (expressed in terms of wavelength) with experimental results for a temperature $T = 1595°\text{K}$. The experimental results are in complete agreement with Planck's formula at all temperatures.

We should remember that Planck did not alter the Boltzmann distribution. "All" he did was to treat the energy of the electromagnetic standing waves, oscillating sinusoidally in time, as a discrete instead of a continuous quantity.

Example 1-4. Derive Planck's expression for the average energy $\bar{\mathscr{E}}$ and also his blackbody spectrum.

►The quantity $\bar{\mathscr{E}}$ is evaluated from the ratio of sums

$$\bar{\mathscr{E}} = \frac{\sum_{n=0}^{\infty} \mathscr{E}P(\mathscr{E})}{\sum_{n=0}^{\infty} P(\mathscr{E})}$$

analogous to the ratio of integrals in (1-21). Sums must be used because with Planck's postulate the energy $\mathscr{E}$ becomes a discrete variable that takes on only the values $\mathscr{E} = 0, h\nu, 2h\nu, 3h\nu, \ldots$. That is, $\mathscr{E} = nh\nu$ where $n = 0, 1, 2, 3, \ldots$. Evaluating the Boltzmann distribution $P(\mathscr{E}) = e^{-\mathscr{E}/kT}/kT$, we have

$$\bar{\mathscr{E}} = \frac{\sum_{n=0}^{\infty} \frac{nh\nu}{kT} e^{-nh\nu/kT}}{\sum_{n=0}^{\infty} \frac{1}{kT} e^{-nh\nu/kT}} = kT\frac{\sum_{n=0}^{\infty} n\alpha e^{-n\alpha}}{\sum_{n=0}^{\infty} e^{-n\alpha}} \qquad \text{where } \alpha = \frac{h\nu}{kT}$$

This, in turn, can be evaluated most easily by noting that

$$-\alpha\frac{d}{d\alpha}\ln\sum_{n=0}^{\infty} e^{-n\alpha} = \frac{-\alpha\frac{d}{d\alpha}\sum_{n=0}^{\infty} e^{-n\alpha}}{\sum_{n=0}^{\infty} e^{-n\alpha}} = \frac{-\sum_{n=0}^{\infty}\alpha\frac{d}{d\alpha}e^{-n\alpha}}{\sum_{n=0}^{\infty} e^{-n\alpha}} = \frac{\sum_{n=0}^{\infty} n\alpha e^{-n\alpha}}{\sum_{n=0}^{\infty} e^{-n\alpha}}$$

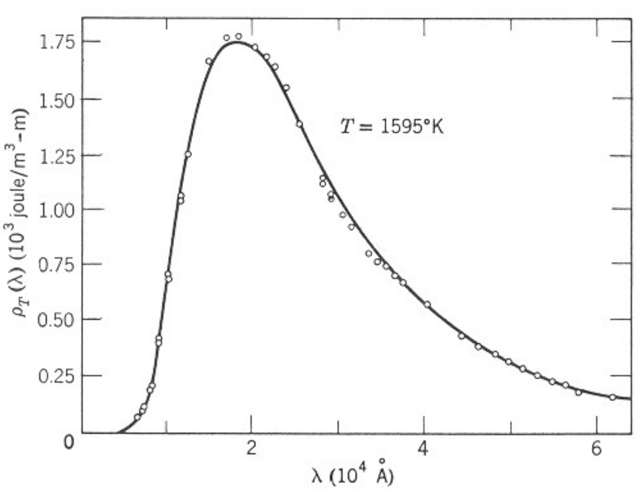

Figure 1-11 Planck's energy density prediction (solid line) compared to the experimental results (circles) for the energy density of a blackbody. The data were reported by Coblentz in 1916 and apply to a temperature of 1595°K. The author remarked in his paper that after drawing the spectral energy curves resulting from his measurements, "owing to eye fatigue it was impossible for months thereafter to give attention to the reduction of the data." The data, when finally reduced, led to a value for Planck's constant of 6.57×10^{-34} joule-sec.

so that

$$\bar{\mathscr{E}} = kT\left(-\alpha \frac{d}{d\alpha} \ln \sum_{n=0}^{\infty} e^{-n\alpha}\right) = -h\nu \frac{d}{d\alpha} \ln \sum_{n=0}^{\infty} e^{-n\alpha}$$

Now

$$\sum_{n=0}^{\infty} e^{-n\alpha} = 1 + e^{-\alpha} + e^{-2\alpha} + e^{-3\alpha} + \cdots$$

$$= 1 + X + X^2 + X^3 + \cdots \qquad \text{where } X = e^{-\alpha}$$

but

$$(1 - X)^{-1} = 1 + X + X^2 + X^3 + \cdots$$

so

$$\bar{\mathscr{E}} = -h\nu \frac{d}{d\alpha} \ln (1 - e^{-\alpha})^{-1}$$

$$= \frac{-h\nu}{(1 - e^{-\alpha})^{-1}} (-1)(1 - e^{-\alpha})^{-2} e^{-\alpha}$$

$$= \frac{h\nu e^{-\alpha}}{1 - e^{-\alpha}} = \frac{h\nu}{e^{\alpha} - 1} = \frac{h\nu}{e^{h\nu/kT} - 1}$$

We have derived (1-26) for the average energy of an electromagnetic standing wave of frequency ν. Multiplying this by (1-12), the number $N(\nu)\,d\nu$ of waves having this frequency derived in Example 1-3, we immediately obtain the Planck blackbody spectrum, (1-27). ◀

Example 1-5. It is convenient in analyzing experimental results, as in Figure 1-11, to express the Planck blackbody spectrum in terms of wavelength λ rather than frequency ν. Obtain $\rho_T(\lambda)$, the wavelength form of Planck's spectrum, from $\rho_T(\nu)$, the frequency form of the spectrum. The quantity $\rho_T(\lambda)$ is defined from the equality $\rho_T(\lambda)\,d\lambda = -\rho_T(\nu)\,d\nu$. The minus sign indicates that, though $\rho_T(\lambda)$ and $\rho_T(\nu)$ are both positive, $d\lambda$ and $d\nu$ have opposite signs. (An increase in frequency gives rise to a corresponding decrease in wavelength.)

▶From the relation $\nu = c/\lambda$ we have $d\nu = -(c/\lambda^2)\,d\lambda$, or $d\nu/d\lambda = -(c/\lambda^2)$, so that

$$\rho_T(\lambda) = -\rho_T(\nu)\frac{d\nu}{d\lambda} = \rho_T(\nu)\frac{c}{\lambda^2}$$

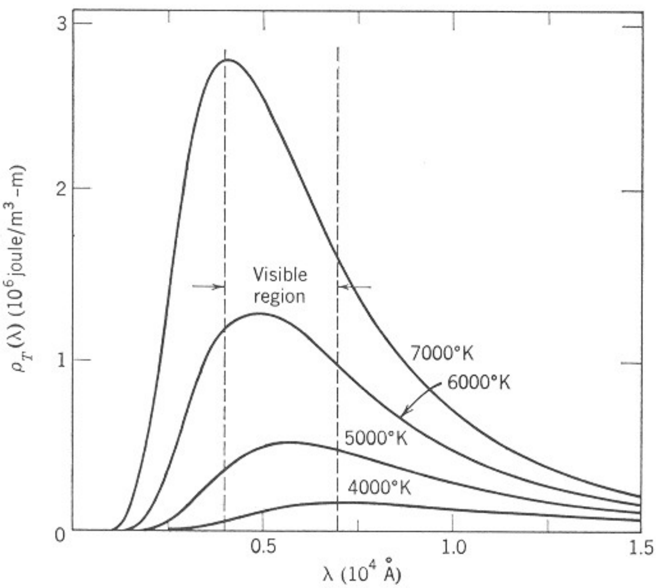

Figure 1-12 Planck's energy density of blackbody radiation at various temperatures as a function of wavelength. Note that the wavelength at which the curve is a maximum decreases as the temperature increases.

If now we set $v = c/\lambda$ in (1-27) for $\rho_T(v)$ we obtain

$$\rho_T(\lambda)\,d\lambda = \frac{8\pi hc}{\lambda^5}\frac{d\lambda}{e^{hc/\lambda kT} - 1} \tag{1-28}$$

In Figure 1-12 we show $\rho_T(\lambda)$ versus λ for several different temperatures. The trend from "red heat" to "white heat" to "blue heat" radiation with rising temperatures becomes clear as the distribution of radiant energy with wavelength is studied for increasing temperatures. ◀

Stefan's law, (1-2), and Wien's displacement law, (1-3), can be derived from the Planck formula. By fitting them to the experimental results we can determine values of the constants h and k. Stefan's law is obtained by integrating Planck's law over the entire spectrum of wavelengths. The radiancy is found to be proportional to the fourth power of the temperature, the proportionality constant $2\pi^5k^4/15c^2h^3$ being identified with σ, Stefan's constant, which has the experimentally determined value 5.67×10^{-8} W/m^2-°K^4. Wien's displacement law is obtained by setting $d\rho(\lambda)/d\lambda = 0$. We find $\lambda_{\max}T = 0.2014hc/k$ and identify the right-hand side of the equation with Wien's experimentally determined constant 2.898×10^{-3} m-°K. Using these two measured values and assuming a value for the speed of light c, we can calculate the values of h and k. Indeed, this was done by Planck, his values agreeing very well with those obtained subsequently by other methods.

1-5 THE USE OF PLANCK'S RADIATION LAW IN THERMOMETRY

The radiation emitted from a hot body can be used to measure its temperature. If total radiation is used, then, from the Stefan-Boltzmann law, we know that the energies emitted by two sources are in the ratio of the fourth power of the temperature. However, it is difficult to measure total radiation from most sources so that we measure instead the radiancy over a finite wavelength band. Here we use the Planck radiation law which gives the radiancy as a function of temperature and wavelength. For monochromatic radiation of wavelength λ the ratio of the spectral intensities emitted by sources at T_2 °K and T_1 °K is given from Planck's law as

$$\frac{e^{hc/\lambda kT_1} - 1}{e^{hc/\lambda T_2} - 1}$$

If T_1 is taken as a standard reference temperature, then T_2 can be determined relative to the standard from this expression by measuring the ratio experimentally. This procedure is used in the International Practical Temperature Scale, where the normal melting point of gold is taken as the standard fixed point, 1068°C. That is, the primary standard *optical pyrometer* is arranged to compare the spectral radiancy from a blackbody at an unknown temperature $T > 1068$°C with a blackbody at the gold point. Procedures must be adopted, and the theory developed, to allow for the practical circumstances that most sources are not blackbodies and that a finite spectral band is used instead of monochromatic radiation.

Most optical pyrometers use the eye as a detector and call for a large spectral bandwidth so that there will be enough energy for the eye to see. The simplest and most accurate type of instrument used above the gold point is the disappearing filament optical pyrometer (see Figure 1-13). The source whose temperature is to be measured is imaged on the filament of the

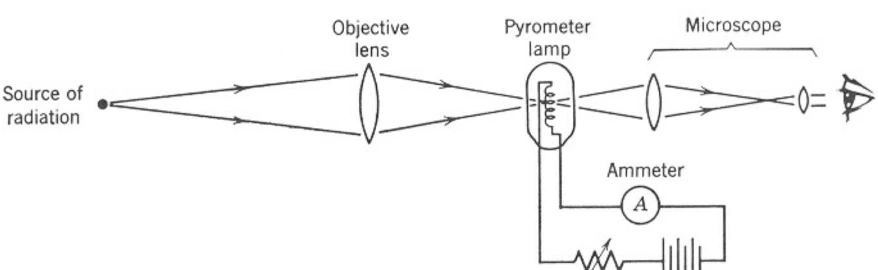

Figure 1-13 Schematic diagram of an optical pyrometer.

pyrometer lamp, and the current in the lamp is varied until the filament seems to disappear into the background of the source image. Careful calibration and precision potentiometers insure accurate measurement of temperature.

A particularly interesting example in the general category of thermometry using blackbody radiation was discovered by Dicke, Penzias, and Wilson in the 1950s. Using a radio telescope operating in the several millimeter to several centimeter wavelength range, they found that a blackbody spectrum of electromagnetic radiation, with a characteristic temperature of about 3°K, is impinging on the earth with equal intensity from all directions. The uniformity in direction indicates that the radiation fills the universe uniformly. Astrophysicists consider these measurements as strong evidence in favor of the so-called *big-bang theory*, in which the universe was in the form of a very dense, and hot, fireball of particles and radiation around 10^{10} years ago. Due to subsequent expansion and the resulting Doppler shift, the temperature of the radiation would be expected to drop by now to something like the observed value of 3°K.

1-6 PLANCK'S POSTULATE AND ITS IMPLICATIONS

Planck's contribution can be stated as a postulate, as follows:

Any physical entity with one degree of freedom whose "coordinate" is a sinusoidal function of time (i.e., executes simple harmonic oscillations) can possess only total energies $\mathscr{E}$ which satisfy the relation

$$\mathscr{E} = nh\nu \qquad n = 0, 1, 2, 3, \ldots$$

where ν is the frequency of the oscillation, and h is a universal constant.

The word coordinate is used in its general sense to mean any quantity which describes the instantaneous condition of the enity. Examples are the length of a coil spring, the angular position of a pendulum bob, and the amplitude of a wave. All these examples happen also to be sinusoidal functions of time.

An energy-level diagram, as shown in Figure 1-14, provides a convenient way of illustrating the behavior of an entity governed by this postulate, and it is also useful in contrasting this behavior with what would be expected on the basis of classical physics. In such a diagram we indicate each of the possible energy states of the entity with a horizontal line. The distance from the line to the zero energy line is proportional to the total energy to which it corresponds. Since the entity may have any energy from zero to infinity according to classical physics, the classical energy-level diagram consists of a continuum of lines extending from zero up. However, the entity executing simple harmonic oscillations can have only one of the discrete total energies $\mathscr{E} = 0, h\nu, 2h\nu, 3h\nu \ldots$ if it obeys Planck's postulate. This is indicated by the discrete set of lines in its energy-level diagram. The energy of the entity obeying Planck's postulate is said to be *quantized*, the allowed energy states are called *quantum states*, and the integer n is called the *quantum number*.

It may have occurred to the student that there are physical systems whose behavior seems to be obviously in disagreement with Planck's postulate. For instance, an ordi-

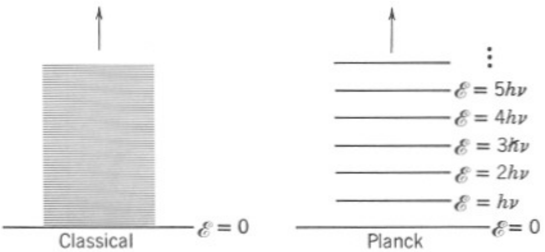

Figure 1-14 *Left:* The allowed energies in a classical system, oscillating sinusoidally with frequency ν, are continuously distributed. *Right:* The allowed energies according to Planck's postulate are discretely distributed since they can only assume the values $nh\nu$. We say that the energy is quantized, n being the quantum number of an allowed quantum state.

nary pendulum executes simple harmonic oscillations, and yet this system certainly appears to be capable of possessing a continuous range of energies. Before we accept this argument, however, we should make some simple numerical calculations concerning such a system.

Example 1-6. A pendulum consisting of a 0.01 kg mass is suspended from a string 0.1 m in length. Let the amplitude of its oscillation be such that the string in its extreme positions makes an angle of 0.1 rad with the vertical. The energy of the pendulum decreases due, for instance, to frictional effects. Is the energy decrease observed to be continuous or discontinuous?

►The oscillation frequency of the pendulum is

$$\nu = \frac{1}{2\pi}\sqrt{\frac{g}{l}} = \frac{1}{2\pi}\sqrt{\frac{9.8\ \text{m/sec}^2}{0.1\ \text{m}}} = 1.6/\text{sec}$$

The energy of the pendulum is its maximum potential energy

$$mgh = mgl(1 - \cos\theta) = 0.01\ \text{kg} \times 9.8\ \text{m/sec}^2 \times 0.1\ \text{m} \times (1 - \cos 0.1)$$
$$= 5 \times 10^{-5}\ \text{joule}$$

The energy of the pendulum is quantized so that changes in energy take place in discontinuous jumps of magnitude $\Delta E = h\nu$, but

$$\Delta E = h\nu = 6.63 \times 10^{-34}\ \text{joule-sec} \times 1.6/\text{sec} = 10^{-33}\ \text{joule}$$

whereas $E = 5 \times 10^{-5}$ joule. Therefore, $\Delta E/E = 2 \times 10^{-29}$. Hence, to measure the discreteness in the energy decrease we need to measure the energy to better than two parts in 10^{29}. It is apparent that even the most sensitive experimental equipment is totally incapable of this energy resolution. ◄

We conclude that experiments involving an ordinary pendulum cannot determine whether Planck's postulate is valid or not. The same is true of experiments on all other macroscopic mechanical systems. The smallness of h makes the graininess in the energy too fine to be distinguished from an energy continuum. Indeed, h might as well be zero for classical systems and, in fact, one way to reduce quantum formulas to their classical limits would be to let $h \to 0$ in these formulas. Only where we consider systems in which ν is so large and/or $\mathscr{E}$ is so small that $\Delta\mathscr{E} = h\nu$ is of the order of $\mathscr{E}$ are we in a position to test Planck's postulate. One example is, of course, the high-frequency standing waves in blackbody radiation. Many other examples will be considered in following chapters.

1-7 A BIT OF QUANTUM HISTORY

In its original form, Planck's postulate was not so far reaching as it is in the form we have given. Planck's initial work was done by treating, in detail, the behavior of the electrons in the walls of the blackbody and their coupling to the electromagnetic radiation within the cavity. This coupling leads to the same factor ν^2 we obtained in (1-12) from the more general arguments due to Rayleigh and Jeans. Through this coupling, Planck related the energy in a particular frequency component of the blackbody radiation to the energy of an electron in the wall oscillating sinusoidally at the same frequency, and he postulated only that the energy of the oscillating particle is quantized. It was not until later that Planck accepted the idea that the oscillating electromagnetic waves were themselves quantized, and the postulate was broadened to include any entity whose single coordinate oscillates sinusoidally.

At first Planck was unsure whether his introduction of the constant h was only a mathematical device or a matter of deep physical significance. In a letter to R. W. Wood, Planck called his limited postulate "an act of desperation." "I knew," he wrote, "that the problem (of the equilibrium of matter and radiation) is of fundamental significance for physics; I knew the formula that reproduces the energy distribution in the normal spectrum; a theoretical interpretation *had* to be found at any cost, no matter how high." For more than a decade Planck tried to fit the quantum idea into classical theory. With each attempt he appeared to retreat from his original boldness, but always he generated new ideas and techniques that quantum theory later adopted. What appears to have finally convinced him of the correctness and deep significance of his quantum hypothesis was its support of the definiteness of the statistical concept of entropy and the third law of thermodynamics.

It was during this period of doubt that Planck was editor of the German research journal *Annalen der Physik*. In 1905 he received Einstein's first relativity paper and stoutly defended Einstein's work. Thereafter he became one of young Einstein's patrons in scientific circles, but he resisted for some time the very ideas on the quantum theory of radiation advanced by Einstein that subsequently confirmed and extended Planck's own work. Einstein, whose deep insight into electromagnetism and statistical mechanics was perhaps unequalled by anyone at the time, saw as a result of Planck's work the need for a sweeping change in classical statistics and electromagnetism. He advanced predictions and interpretations of many physical phenomena which were later strikingly confirmed by experiment. In the next chapter we turn to one of these phenomena and follow another road on the way to quantum mechanics.

QUESTIONS

1. Does a blackbody always appear black? Explain the term blackbody.

2. Pockets formed by coals in a coal fire seem brighter than the coals themselves. Is the temperature in such pockets appreciably higher than the surface temperature of an exposed glowing coal?

3. If we look into a cavity whose walls are kept at a constant temperature no details of the interior are visible. Explain.

4. The relation $R_T = \sigma T^4$ is exact for blackbodies and holds for all temperatures. Why is this relation not used as the basis of a definition of temperature at, for instance, 100°C?

5. A piece of metal glows with a bright red color at 1100°K. At this temperature, however, a piece of quartz does not glow at all. Explain. (Hint: Quartz is transparent to visible light.)

6. Make a list of distribution functions commonly used in the social sciences (e.g., distribution of families with respect to income). In each case, state whether the variable whose distribution is described is discrete or continuous.

7. In (1-4) relating spectral radiancy and energy density, what dimensions would a proportionality constant need to have?

8. What is the origin of the ultraviolet catastrophe?

9. The law of equipartition of energy requires that the specific heat of gases be independent of the temperature, in disagreement with experiment. Here we have seen that it leads to the Rayleigh-Jeans radiation law, also in disagreement with experiment. How can you relate these two failures of the equipartition law?

10. Compare the definitions and dimensions of spectral radiancy $R_T(\nu)$, radiancy R_T, and energy density $\rho_T(\nu)$.

11. Why is optical pyrometry commonly used above the gold point and not below it? What objects typically have their temperatures measured in this way?

12. Are there quantized quantities in classical physics? Is energy quantized in classical physics?

13. Does it make sense to speak of charge quantization in physics? How is this different from energy quantization?

14. Elementary particles seem to have a discrete set of rest masses. Can this be regarded as quantization of mass?

15. In many classical systems the allowed frequencies are quantized. Name some of the systems. Is energy quantized there too?

16. Show that Planck's constant has the dimensions of angular momentum. Does this necessarily suggest that angular momentum is a quantized quantity?

17. For quantum effects to be everyday phenomena in our lives, what would be the minimum order of magnitude of h?

18. What, if anything, does the 3°K universal blackbody radiation tell us about the temperature of outer space?
19. Does Planck's theory suggest quantized atomic energy states?
20. Discuss the remarkable fact that discreteness in energy was first found in analyzing a continuous spectrum emitted by interacting atoms in a solid, rather than in analyzing a discrete spectrum such as is emitted by an isolated atom in a gas.

PROBLEMS

1. At what wavelength does a cavity at 6000°K radiate most per unit wavelength?
2. Show that the proportionality constant in (1-4) is $4/c$. That is, show that the relation between spectral radiancy $R_T(\nu)$ and energy density $\rho_T(\nu)$ is $R_T(\nu)\,d\nu = (c/4)\rho_T(\nu)\,d\nu$.
3. Consider two cavities of arbitrary shape and material, each at the same temperature T, connected by a narrow tube in which can be placed color filters (assumed ideal) which will allow only radiation of a specified frequency ν to pass through. (a) Suppose at a certain frequency ν', $\rho_T(\nu')\,d\nu$ for cavity 1 was greater than $\rho_T(\nu')\,d\nu$ for cavity 2. A color filter which passes only the frequency ν' is placed in the connecting tube. Discuss what will happen in terms of energy flow. (b) What will happen to their respective temperatures? (c) Show that this would violate the second law of thermodynamics; hence prove that all blackbodies at the same temperature must emit thermal radiation with the same spectrum independent of the details of their composition.
4. A cavity radiator at 6000°K has a hole 10.0 mm in diameter drilled in its wall. Find the power radiated through the hole in the range 5500–5510 Å. (Hint: See Problem2.)
5. (a) Assuming the surface temperature of the sun to be 5700°K, use Stefan's law, (1-2), to determine the rest mass lost per second to radiation by the sun. Take the sun's diameter to be 1.4×10^9 m. (b) What fraction of the sun's rest mass is lost each year from electromagnetic radiation? Take the sun's rest mass to be 2.0×10^{30} kg.
6. In a thermonuclear explosion the temperature in the fireball is momentarily 10^7 °K. Find the wavelength at which the radiation emitted is a maximum.
7. At a given temperature, $\lambda_{max} = 6500$ Å for a blackbody cavity. What will λ_{max} be if the temperature of the cavity walls is increased so that the rate of emission of spectral radiation is doubled?
8. At what wavelength does the human body emit its maximum temperature radiation? List assumptions you make in arriving at an answer.
9. Assuming that λ_{max} is in the near infrared for red heat and in the near ultraviolet for blue heat, approximately what temperature in Wien's displacement law corresponds to red heat? To blue heat?
10. The *average* rate of solar radiation incident per unit area on the earth is 0.485 cal/cm^2-min (or 338 W/m^2). (a) Explain the consistency of this number with the solar constant (the solar energy falling per unit time at normal incidence on a unit area) whose value is 1.94 cal/cm^2-min (or 1353 W/m^2). (b) Consider the earth to be a blackbody radiating energy into space at this same rate. What surface temperature would the earth have under these circumstances?
11. Attached to the roof of a house are three solar panels, each 1 m × 2 m. Assume the equivalent of 4 hrs of normally incident sunlight each day, and that all the incident light is absorbed and converted to heat. How many gallons of water can be heated from 40°C to 120°C each day?
12. Show that the Rayleigh-Jeans radiation law, (1-17), is not consistent with the Wien displacement law $\nu_{max} \propto T$, (1-3a), or $\lambda_{max}T = \text{const}$, (1-3b).
13. We obtain ν_{max} in the blackbody spectrum by setting $d\rho_T(\nu)/d\nu = 0$ and λ_{max} by setting $d\rho_T(\lambda)/d\lambda = 0$. Why is it not possible to get from $\lambda_{max}T = \text{const}$ to $\nu_{max} = \text{const} \times T$ simply by using $\lambda_{max} = c/\nu_{max}$? That is, why is it wrong to assume that $\nu_{max}\lambda_{max} = c$, where c is the speed of light?
14. Consider the following numbers: 2, 3, 3, 4, 1, 2, 2, 1, 0 representing the number of hits garnered by each member of the Baltimore Orioles in a recent outing. (a) Calculate

directly the average number of hits per man. (b) Let x be a variable signifying the number of hits obtained by a man, and let $f(x)$ be the number of times the number x appears. Show that the average number of hits per man can be written as

$$\bar{x} = \frac{\sum_0^4 xf(x)}{\sum_0^4 f(x)}$$

(c) Let $p(x)$ be the probability of the number x being attained. Show that $\bar{x}$ is given by

$$\bar{x} = \sum_0^4 xp(x)$$

15. Consider the function

$$f(x) = \frac{1}{10}(10 - x)^2 \qquad 0 \leq x \leq 10$$

$$f(x) = 0 \qquad \text{all other } x$$

(a) From

$$\bar{x} = \frac{\int_{-\infty}^{\infty} xf(x)\,dx}{\int_{-\infty}^{\infty} f(x)\,dx}$$

find the average value of x. (b) Suppose the variable x were discrete rather than continuous. Assume $\Delta x = 1$ so that x takes on only integral values $0, 1, 2, \ldots, 10$. Compute $\bar{x}$ and compare to the result of part (a). (Hint: It may be easier to compute the appropriate sum directly rather than working with general summation formulas.) (c) Compute $\bar{x}$ for $\Delta x = 5$, i.e. $x = 0, 5, 10$. Compare to the result of part (a). (d) Draw analogies between the results obtained in this problem and the discussion of Section 1-4. Be sure you understand the roles played by $\bar{\mathscr{E}}$, $\Delta\mathscr{E}$, and $P(\mathscr{E})$.

16. Using the relations $P(\mathscr{E}) = e^{-\mathscr{E}/kT}/kT$ and $\int_0^\infty P(\mathscr{E})\,d\mathscr{E} = 1$, evaluate the integral of (1-21) to deduce (1-22), $\bar{\mathscr{E}} = kT$.

17. Use the relation $R_T(\nu)\,d\nu = (c/4)\rho_T(\nu)\,d\nu$ between spectral radiancy and energy density, together with Planck's radiation law, to derive Stefan's law. That is, show that

$$R_T = \int_0^\infty \frac{2\pi h}{c^2} \frac{\nu^3\,d\nu}{e^{h\nu/kT} - 1} = \sigma T^4$$

where $\sigma = 2\pi^5 k^4/15c^2h^3$.

$$\left(\text{Hint: } \int_0^\infty \frac{q^3\,dq}{e^q - 1} = \frac{\pi^4}{15}\right)$$

18. Derive the Wien displacement law, $\lambda_{\max} T = 0.2014\,hc/k$, by solving the equation $d\rho(\lambda)/d\lambda = 0$. (Hint: Set $hc/\lambda kT = x$ and show that the equation quoted leads to $e^{-x} + x/5 = 1$. Then show that $x = 4.965$ is the solution.)

19. To verify experimentally that the 3°K universal background radiation accurately fits a blackbody spectrum, it is decided to measure $R_T(\lambda)$ from a wavelength below $\lambda_{\max}$ where its value is $0.2R_T(\lambda_{\max})$ to a wavelength above $\lambda_{\max}$ where its value is again $0.2R_T(\lambda_{\max})$. Over what range of wavelength must the measurements be made?

20. Show that, at the wavelength $\lambda_{\max}$, where $\rho_T(\lambda)$ has its maximum

$$\rho_T(\lambda_{\max}) = 170\pi(kT)^5/(hc)^4$$

21. Use the result of the preceding problem to find the two wavelengths at which $\rho_T(\lambda)$ has a value one-half the value at λ_{max}. Give answers in terms of λ_{max}.

22. A tungsten sphere 2.30 cm in diameter is heated to 2000°C. At this temperature tungsten radiates only about 30% of the energy radiated by a blackbody of the same size and temperature. (a) Calculate the temperature of a perfectly black spherical body of the same size that radiates at the same rate as the tungsten sphere. (b) Calculate the diameter of a perfectly black spherical body at the same temperature as the tungsten sphere that radiates at the same rate.

23. (a) Show that about 25% of the radiant energy in a cavity is contained within wavelengths zero and λ_{max}; i.e., show that

$$\frac{\displaystyle\int_0^{\lambda_{max}} \rho_T(\lambda)\,d\lambda}{\displaystyle\int_0^{\infty} \rho_T(\lambda)\,d\lambda} \simeq \frac{1}{4}$$

(Hint: $hc/\lambda_{max}kT = 4.965$; hence Wien's approximation is fairly accurate in evaluating the integral in the numerator above.) (b) By what percent does Wien's approximation used over the entire wavelength range overestimate or underestimate the integrated energy density?

24. Find the temperature of a cavity having a radiant energy density at 2000 Å that is 3.82 times the energy density at 4000 Å.