A $k$-coloring of (the vertex set of) a graph $G$ is a function $c : V(G) \to \{1, 2, \ldots, k\}$ such that $c(u) \neq c(v)$ whenever $u$ is adjacent to $v$. If a $k$-coloring of $G$ exists, then $G$ is called $k$-colorable. The chromatic number of $G$ is defined as

$$\chi(G) = \min \{k : G \text{ is } k\text{-colorable.}\}$$

When $\chi(G) = k$, $G$ is called $k$-chromatic.

6.1 Lower bounds of the chromatic number

Let $\omega(G)$ be the clique number of $G$, that is, the maximal number of vertices in a complete subgraph of $G$. Obviously, we have

$$\chi(G) \geq \omega(G) \quad (6.1)$$

Given a $k$-coloring of $G$, the vertices being colored with the same color form an independent set. Let $G$ be a graph with $n$ vertices and $c$ a $k$-coloring of $G$. We define

$$V_i = \{v : c(v) = i\}$$

for $i = 1, 2, \ldots, k$. Each $V_i$ is an independent set. Let $\alpha(G)$ be the independence number of $G$, we have

$$|V_i| \leq \alpha(G).$$

Since

$$n = |V_1| + |V_2| + \cdots + |V_k| \leq k \cdot \alpha(G) = \chi(G) \alpha(G),$$

we have

$$\chi(G) \geq \frac{n}{\alpha(G)}. \quad (6.2)$$

These bounds can be very far off. For (6.2), there are graphs with $\frac{n}{\alpha(G)} < 2$ but $\chi(G)$ arbitrarily large.

(6.1) is not much better either. There are graphs with $\chi(G) > \omega(G)$, for example, $C_5$. $\chi(C_5) = 3$ and $\omega(C_5) = 2$. Let $G$ and $H$ be two graphs. We define the join of $G$ and $H$, $G \oplus H$ to be the graph obtained by joining every vertex in $G$ with
every vertex in $H$. Clearly, $\chi(G \oplus H) = \chi(G) + \chi(H)$ and $\omega(G \oplus H) = \omega(G) + \omega(H)$. So $\chi(C_5 \oplus C_5) = 6$ and $\omega(C_5 \oplus C_5) = 4$. Let $G_1 = C_5 \oplus C_5$ and $G_p = G_{p-1} \oplus G_{p-1}$, we have

$$
\begin{align*}
\chi(G_1) &= 6, \omega(G_1) = 4 \\
\chi(G_2) &= 12, \omega(G_2) = 8 \\
\chi(G_3) &= 24, \omega(G_3) = 16 \\
\vdots \\
\chi(G_p) &= 3 \cdot 2^p, \omega(G_p) = 2^{p+1} \\
\chi(G_p) - \omega(G_p) &= 2^p
\end{align*}
$$

The difference between $\chi(G)$ and $\omega(G)$ can be arbitrarily large. Furthermore, even when $\omega(G) = 2$, we can have $\chi(G)$ arbitrarily large.

When $\omega(G) = 2$, $G$ is called triangle-free.

6.1.1 Tutte’s construction of triangle-free $k$-chromatic graphs

We construct $G_k$ recursively.

$G_2 = K_2$.

Let $n_k = |G_k|$, the number of vertices in $G_k$.

Suppose that $G_{k-1}$ has been built. Let $I_k$ be an independent set of $(n_{k-1} - 1) (k - 1) + 1$ vertices. For every $(k - 1)$-subset of $I_k$, we take a copy of $G_{k-1}$ and match the vertices in that copy to the vertices in the $(k - 1)$-subsets. The result is $G_k$.

Example: $G_3$.

Claim 6.1.1 $G_k$ is triangle-free.

Proof. Prove by induction on $k$.

$G_2$ is obviously triangle-free. Assume that $G_{k-1}$ is triangle-free. No vertex in $I_k$ can be in a triangle: Let $x$ be a vertex in $I_k$. All vertices that are adjacent to $x$ are in different copies of $G_{k-1}$. So no two neighbors of $x$ are adjacent. Since there are no triangles in the copies of $G_{k-1}$, there is no triangle in $G_k$.

Claim 6.1.2 $\chi(G) = k$.

Proof. By induction on $k$. Since it is easy to see that $G_k$ is $k$-colorable (color all the copies of $G_{k-1}$ with colors $1, 2, \ldots, k-1$ and color the vertices in $I_k$ with color $k$), we only need to show that $G_k$ is not $(k-1)$-colorable. Suppose that $G_k$ is $(k-1)$-colorable and $c$ is a $(k-1)$-coloring of $G_k$. By the pigeonhole principle, there are at least $n_{k-1}$ vertices that are colored with the same color. Then the copy of $G_{k-1}$ must be colored with $k - 2$ colors. Contradiction.
6.2 Upper bounds

6.2.1 The greedy algorithm
Let the vertices of a graph \( G \) be \( v_1, v_2, ..., v_n \) and the maximum degree of \( G \) be \( \Delta \). For \( i = 1, 2, ..., n \) we color \( v_i \) with the smallest number that is available, i.e., the number that has not been used on the vertices in \( \{ v_1, v_2, ..., v_{i-1} \} \) that are adjacent to \( v_i \). Since there are at most \( \Delta \) vertices that are adjacent to \( v_i \), this algorithm will not use more than \( \Delta + 1 \) colors. Therefore we have

\[
\chi(G) \leq \Delta(G) + 1 \tag{6.3}
\]

(6.3) can be pretty bad. The graph \( K_{1,p} \) is called a star. It has \( \Delta(G) = p \) and \( \chi(G) = 2 \). However, for a complete graph \( K_n \) and an odd cycle, we have \( \chi(G) = \Delta(G) + 1 \). Therefore, (6.3) cannot be improved in general.

**Theorem 6.2.2** Suppose that in every subgraph \( H \) of \( G \) there is a vertex with degree at most \( \delta \) in \( H \).

\[
\chi(G) \leq \delta + 1
\]

**Proof.** There is a vertex with degree at most \( \delta \) in \( G \). Label that vertex \( v_n \). There is a vertex with degree at most \( \delta \) in \( G - v_n \). Label that vertex \( v_{n-1} \). Label the vertex with degree at most \( \delta \) in \( G - \{ v_n, v_{n-1} \} \). Continue to label all the vertices in \( G \) as \( v_1, v_2, ..., v_n \). Apply the greedy algorithm according to this labeling. At each step, the vertex we are going to color is adjacent to at most \( \delta \) vertices that are already colored. Therefore, \( \delta + 1 \) colors will be enough.

For a star \( K_{1,p} \) Theorem 6.2.1 provides the upper bound \( \chi(G) \leq p + 1 \). Since every subgraph of \( K_{1,p} \) has a vertex of degree at most one, Theorem 6.2.2 gives the upper bound \( \chi(G) \leq 2 \).

**Definition 6.2.3** A graph is called a wheel \( (W_n) \) if it is obtained by joining a new vertex to every vertex in a cycle \( C_n \). The wheel is called odd (or even) if \( n \) is odd (or even).

In the figure that follows is the wheel \( W_7 \). It has maximum degree 7. All its subgraphs contain a vertex of degree at most three. Theorem 6.2.1 gives an upper bound \( \chi(G) \leq 8 \) and Theorem 6.2.2 gives \( \chi(G) \leq 4 \). Actually we can see that \( \chi(G) = 4 \).
6.2.2 Brook’s Theorem

Theorem 6.2.2 is difficult to apply in general. Another way to improve (6.3) is the following theorem.

Theorem 6.2.4 (Brook’s Theorem) Suppose that $\Delta(G) = k$. $\chi(G) \leq k$ unless $G$ contains a $K_{k+1}$ or $k = 2$ and $G$ is an odd cycle.

6.3 Planar graphs and the Four Colour Theorem

One earlier focus of graph theory is the Four Colour Problem:

Given a map of countries such that every country is a continuous piece, is it always possible to use four colors to color the countries such that any two countries with a common boundary are colored with different colors?

It is easy to see that three colors would not be enough and no one had found an example of a map that needed more than four colors. However, no one can prove that such examples do not exist for a long time. It is until 1976 with the help of thousands of hours of computer time, a proof was found.

6.3.1 Dual graphs

A map can be represented by a graph such that each country is represented by a vertex and two countries share a boundary if and only if the corresponding vertices are adjacent. That graph is called a dual graph of the map. A coloring of the map satisfying the condition of the four color problem corresponding to a coloring of the vertices of the dual graph. A graph is planar if it can be drawn (or embedded) in the plane such that no two edges crossing each other. The four color problem is equivalent to

If $G$ is a planar graph, is it true that $\chi(G) \leq 4$?

6.3.2 Euler’s formula

We first study some properties of the planar graphs. The most important property is the Euler’s formula:

Let $G$ be a connected planar graph and $v, e, r$ the numbers of vertices, edges and regions of $G$ respectively.

$$v - e + r = 2.$$

The formula is easy to prove by induction on $v$.

Since every region is bounded by at least three edges and an edge can be used twice in the boundaries, we have

$$3r \leq 2e$$
or

$$r \leq \frac{2e}{3}.$$
Planar graphs and the Four Colour Theorem

Substitute this into (6.4), we have

\[
v - e + \frac{2e}{3} \geq 2 \\
v \geq \frac{1}{3}e + 2 \\
e \leq 3v - 6.
\] (6.5)

**Corollary 6.3.1** $K_5$ is not planar.

**Corollary 6.3.2** $K_{3,3}$ is not planar.

(6.5) cannot be used directly to prove Corollary 6.3.2. We need a stronger version of (6.5) for bipartite graphs. Since a bipartite graph does not contain any triangles, a region is bounded by at least 4 edges. For all bipartite graphs, we have

\[
4r \leq 2e \\
r \leq \frac{1}{2}e.
\]

That will imply

\[
v - e + \frac{1}{2}e \geq 2 \\
v \geq \frac{1}{2}e + 2 \\
e \leq 2v - 4.
\] (6.6)

(6.6) implies Corollary 6.3.2.

*Subdividing* an edge or performing an *elementary subdivision* of an edge $uv$ of a graph $G$ is the operation of deleting $uv$ and adding a path $u, w, v$ through a new vertex $w$. A *subdivision* of $G$ is a graph obtained from $G$ by a sequence of elementary subdivisions, turning edges into paths through new vertices of degree 2. Subdividing edges does not affect planarity. Therefore any subdivision of $K_5$ or $K_{6,6}$ is not planar either. Kuratowski [1930] proved that $G$ is planar if and only if $G$ contains no subdivision of $K_5$ or $K_{3,3}$. However, this condition is not easy to check for large graphs.

**6.3.3 Planarity testing**

There are more efficient algorithms to test whether a graph $G$ is planar. The output of these algorithm is either a drawing of $G$ in the plane (when $G$ is planar) or the statement that $G$ is not planar. These algorithm all use the concept of $H$-bridge.

If $H$ is a subgraph of $G$, and $H$-bridge of $G$ is either (1) an edge not in $H$ whose endpoint are in $H$ or (2) a component of $G - V(H)$ together with the edges (and vertices of attachment) that connect it to $H$. The $H$-bridges are the “pieces” that must be added to an embedding of $H$ to obtain an embedding of $G$. 
Graph coloring

Given a cycle $H$ in a graph $G$, two $H$-bridges $A$, $B$ conflict if they have three common vertices of attachment or if there are four vertices $v_1, v_2, v_3, v_4$ in cyclic order on $H$ such that $v_1, v_3$ are vertices of attachment of $A$ and $v_2, v_4$ are vertices of attachment of $B$. In the latter case, we say the $H$-bridges cross. The conflict graph of $H$ is a graph whose vertices are the $H$-bridges of $G$, with conflicting $H$-bridges adjacent.

**Theorem 6.3.3** (Tutte 1958) A graph $G$ is planar if and only if for every cycle $C$ in $G$, the conflict graph of $C$ is bipartite.

It is easy to see that why for a planar graph $G$, the conflict graph must be bipartite. In a planar embedding of $G$, all the $H$-bridges are either inside of $C$ or outside of $C$. The ones on the same side cannot be conflicting.

**Example 6.3.4** The conflict graph of a $C_5$ in $K_5$.

![Graph of $C_5$ in $K_5$ and its conflict graph.]

**Example 6.3.5** The conflict graph of a $C_6$ in $K_{6,6}$.

![Graph of $K_{3,3}$, another drawing of $K_{3,3}$, and its conflict graph.]

There are linear-time planarity-testing algorithms due to Hopcroft and Tarjan (1974) and to Booth and Luecker (1976), but these are very complicated. An earlier algorithm due to Demoucron, Malgrange and Pertuiset (1964) is much simpler. It is
not linear but runs in polynomial time. The idea is that if a planar embedding of \( H \) can be extended to a planar embedding of \( G \), then in that extension every \( H \)-bridge of \( G \) appears inside a single face of \( H \). The algorithm builds increasingly larger plane subgraphs \( H \) of \( G \) that can be extended to an embedding of \( G \) if \( G \) is planar. We want to enlarge \( H \) by making small decisions that won’t lead to trouble.

The basic enlargement step is the following. (1) Choose a face \( F \) that can accept an \( H \)-bridge \( B \); a necessary condition for \( F \) to accept \( B \) is that its boundary contain all vertices of attachment of \( B \). (2) Although we do not know the best way to embed \( B \) in \( F \), each particular path in \( B \) between vertices of attachment by itself has only one way to be added across \( F \), so we add a single such path. (3) If there is an \( H \)-bridge that is not acceptable to any face of \( H \), then \( G \) is not planar.

**Example 6.3.6**

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
5 & 6 & 7 & 8
\end{array}
\]

The first subgraph \( H \) used is the cycle 1, 2, 3, 4, 5. Then the path 3, 7, 6, 5....

**Example 6.3.7**

For the cycle 1, 2, 3, 4, 8, 7, 6, 5, the edges 14, 27, 36 are all crossing. The graph is nonplanar.
6.3.4 The chromatic number of a planar graph

(6.5) implies that for a planar graph $G$, the degree sum has the upper bound

$$\sum_{v} \deg(v) = 2e \leq 6v - 12.$$  

There is a vertex with degree at most 5. Since every subgraph of a planar graph is also a planar graph, every subgraph of $G$ has a vertex with degree at most 5. By Theorem 6.2.2, $\chi(G) \leq 6$. So the six color theorem is easy to proof. With a little of more work, we can prove the five color theorem:

**Theorem 6.3.8** Every planar graph $G$ has chromatic number at most five.

**Proof.** Suppose that the theorem is not true. Let $G$ be the smallest planar graph (in terms of number of vertices) that cannot be colored with five colors. Let $v$ be a vertex in $G$ that has the minimum degree. We know that $\deg(v) \leq 5$.

Case 1: $\deg(v) \leq 4$. $G - v$ can be colored with five colors. There are at most colors have been used on the neighbors of $v$. There is at one color available for $v$. So $G$ can be colored with five colors, a contradiction.

Case 2: $\deg(v) = 5$. $G - v$ can be colored with five colors. If two of the neighbors of $v$ are colored with the same color, then there is a color available for $v$. So we may assume that all the vertices that are adjacent to $v$ are colored with colors 1, 2, 3, 4, 5 in the clockwise order. We will call these vertices $v_1, v_2, ..., v_5$. Consider all the vertices being colored with colors 1 or 3 (and all the edges among them). If this subgraph $G$ is disconnect and $v_1$ and $v_3$ are in different components, then we can switch the colors 1 and 3 in the component with $v_1$. This will still be a 5-coloring of $G - v$. Furthermore, $v_1$ is colored with color 3 in this new 5-coloring and $v_3$ is still colored with color 3. Color 1 would be available for $v$, a contradiction. Therefore $v_1$ and $v_3$ must be in the same component in that subgraph, i.e., there is a path from $v_1$ to $v_3$ such that every vertices on this path is colored with either color 1 or color 3. Similarly, there is a path from $v_2$ to $v_4$ such that every vertices on that path has either color 2 or color 4. These two paths must intersect even though they have no vertex in common. Then there must be two edges crossing each other. This contradiction concludes the proof. ■
6.4 Colour critical graphs

A graph $G$ is (colour) critical if $\chi (H) < \chi (G)$ for every proper subgraph $H$ of $G$. $G$ is said to be $k$-critical if $G$ is critical and $\chi (G) = k$.

Small cases: 1-critical graph. 2-critical graph is an edge. 3-critical graphs are the odd cycles.

**Theorem 6.4.1** If $G$ is a $k$-critical graph, then every vertex in $G$ has degree at least $k - 1$.

*Proof:* Assume that $G$ has a vertex $v$ with degree at most $k - 2$. $G - v$ can be coloured with $k - 1$ colours. At least one of these colours is not used on the neighbors of $v$. Therefore there is at least one colour available for $v$. Then $G$ can be coloured with $k - 1$ colours. This is a contradiction. \(\square\)

**Theorem 6.4.2** (Kainen*) If $G$ is a $k$-critical graph and $X, Y$ is a partition of $V (G)$, then there are at least $k - 1$ edges between $X$ and $Y$.

*Proof:* Assume that there are at most $k - 2$ edges between $X$ and $Y$. We will show that this will lead to a contradiction. Our proof uses Lemma 5.1.7 of the matching theory. Since $G$ is $k$-critical, both $X$ and $Y$ can be colored with $k - 1$ colours. Let the colour classes be $X_1, X_2, \ldots, X_{k-1}$ and $Y_1, Y_2, \ldots, Y_{k-1}$ respectively. We construct a bipartite graph $H$ by letting $V (H) = \{X_1, \ldots, X_{k-1}, Y_1, \ldots, Y_{k-1}\}$ and $X_i Y_j$ is an edge in $H$ if and only if there is no edge joining a vertex in $X_i$ with a vertex in $Y_j$ in $G$. $H$ has at least $(k - 1)^2 - (k - 2) = k^2 - 3k + 3$ edges. Since every vertex has degree at most $k - 1$ therefore covering at most $k - 2$ edges, $k - 2$ vertices can only cover at most $(k - 2) (k - 1) = k^2 - 3k + 2$ edges. Therefore at least $k - 1$ vertices are needed to cover all the edges. By Lemma 5.1.7, $H$ has a matching of $k - 1$ edges which is a complete matching of $H$. If $X_i$ and $Y_j$ are matched, there is no edge between vertices in $X_i$ and vertices in $Y_j$. $X_i \cup Y_j$ is an independent set in $G$. Therefore $V (G)$ can be partitioned into $k - 1$ independent sets. That means that $G$ can be coloured with $k - 1$ colours. This contradiction shows that there must be at least $k - 1$ edges between $X$ and $Y$. \(\square\)

6.4.1 Edge density of 4-critical graphs and Constructions of critical graphs

**Join of two critical graphs.**

**Theorem 6.4.3** Let $G$ be a $k$-critical graph and $H$ a $l$-critical graph. The join of $G$ and $H$, $G \oplus H$ is a $(k + l)$-critical graph.

*Proof:* We have seen that $\chi (G \oplus H) = k + l$. We need to show that $\chi (G \oplus H - e) \leq k + l - 1$ for every edge $e$ in $G \oplus H$.

---

*in D. West: Introduction to Graph Theory, Prentice Hall, 1996.*
Case 1: $e$ is an edge in $G$. We can colour $H$ with colours $1, 2, \ldots, l$ and $G - e$ with $k - 1$ colours: $l + 1, l + 2, \ldots, l + k - 1$.

Case 2: $e$ is an edge in $H$. We can colour $G$ with colours $1, 2, \ldots, k$ and $H - e$ with $l - 1$ colours: $k + 1, k + 2, \ldots, k + l - 1$.

Case 3: $e$ is an edge joining $u \in V(G)$ and $v \in u(G)$. Since $G - u$ can be coloured with $k - 1$ colours, $G$ can be coloured with colours $1, 2, \ldots, k$ such that $u$ is the only vertex being coloured with colour 1. Similarly, $H$ can be coloured with $l$ colours $1, k + 1, k + 2, \ldots, k + l - 1$ such that $v$ is the only vertex being coloured with colour 1. This is a $k + l - 1$ colouring of $G \oplus H - e$.

The 4-critical graphs that we can construct using joins of two critical graphs:

- $K_2 \oplus K_2 = K_4$.
- $K_1 \oplus C_{2l+1} = W_{2l+1}$.

In an odd wheel, there are $2l + 2$ vertices and $4l + 2$ edges. $|E| = 2|V| - 2$.

**Conjecture 6.4.4** (Gallai, 1964) If $G$ is a planar 4-critical graph, then $|E(G)| \leq 2|V(G)| - 2$.

Gallai’s conjecture implies an older conjecture of Dirac:

**Conjecture 6.4.5** (Dirac, 1957) If $G$ is a planar 4-critical graph, then $G$ has a vertex with degree three.

However, both conjectures have been disproved by a graph discovered by Koester\(^\dagger\) in 1985:

The Koester’s graph has 40 vertices and 80 edges. Every vertex has degree four.

Grunbaum\(^\ddagger\) used Hajos’s construction to show that there are planar 4-critical graphs with edge densities (that is the number of edges divided by the number of vertices) arbitrarily close to $79/39 = 2.02564\ldots$. Grunbaum asked the question of determining the maximum edge density of planar 4-critical graphs.

---


\(^\ddagger\)Grunbaum: The edge density of 4-critical planar graphs, *Combinatorica*, 8 (1988), 137-139.
**The Hajos's construction**

Let $G$ and $H$ be two graphs. Let $uu'$ be an edge in $G$ and $vv'$ be an edge in $H$. The graph $G \triangle H$ is obtained by identifying $u$ and $v$, deleting the edges $uu'$ and $vv'$ and adding an edge $u'v'$.

**Theorem 6.4.6** If both $G$ and $H$ are $k$-critical graphs ($k \geq 3$), then $G \triangle H$ is a $k$-critical graph.

**Proof:** First we show that $\chi (G \triangle H) = k$.

(i) $G \triangle H$ cannot be coloured with $k - 1$ colours. $G - uu'$ can be coloured with $k - 1$ colours. In such a colouring, $u$ and $u'$ must be coloured with the same colour (otherwise this would be a $(k - 1)$-colouring of $G$ as well). If $G \triangle H$ were coloured with $k - 1$ colours, then $u'$ has the same colour as $u = v$ which has the same colour as $v'$. Since there is an edge joining $u'$ and $v'$, this impossible.

(ii) $G \triangle H$ can be coloured with $k$ colours. We can colour $G$ with $k$ colours such that $u$ is coloured with colour 1 and $u'$ is coloured with colour 2. Also we can colour $H$ with $k$ colours such that $v$ is coloured with colour 1 and $v'$ is coloured with colour 3. This is a $k$-colouring of $G \triangle H$.

Then we show that $\chi (G \triangle H - e)$ can be coloured with $k - 1$ colours for every edge $e$ in $G \triangle H$.

Case 1: $e$ is in $G$. $G - e$ can be coloured with $k - 1$ colours such that $u$ has colour 1 and $u'$ has colour 2. $H - vv'$ can be coloured $k - 1$ colours such that $v$ and $v'$ both are coloured with colour 1. Combining these, we have a $(k - 1)$-colouring of $G \triangle H - e$.

Case 2: $e$ is in $H$. Similar to Case 1.

Case 3: $e = u'v'$. $G - uu'$ can be coloured with $k - 1$ colours such that both $u$ and $u'$ are coloured with colour 1. $H - vv'$ can be coloured with $k - 1$ colours such that both $v$ and $v'$ are coloured with colour 1. Combining these, we have a $(k - 1)$-colouring of $G - u'v'$.

Let $G_1$ be the Koester’s graph and $G_{i+1} = G_i \triangle G_1$. We have $|E(G_{i+1})| = |E(G_i)| + 79$ and $|V(G_{i+1})| = |V(G_i)| + 39$. Therefore $|E(G_n)| = 79(n - 1) + 80$ and $|V(G_n)| = 39(n - 1) + 40$. The edge density of $G_n$ is

$$\frac{79(n - 1) + 80}{39(n - 1) + 40} \to \frac{79}{39} \text{ as } n \to \infty.$$ 

Abbott and Zhou\(^\S\) used a variation of the Hajos’s construction to show that the edge density of a planar 4-critical graph can be arbitrarily close to $39/19 = 2.05263 \ldots$

The general Hajos’s construction
Let $G$ and $H$ be two graphs. Suppose that $u_1, u_2, ..., u_k$ generate a complete subgraph in $G$ and $v_1, v_2, ..., v_k$ generate a complete subgraph in $H$. $u_i u'$ is an edge in $G$ and $v_i v'$ is an edge in $H$ for some $i$. The graph $G \nabla H$ is obtained by identifying $u_1$ with $v_1$, $u_2$ with $v_2$, ..., $u_k$ with $v_k$, deleting the edges $u_i u'$ and $v_i v'$ and adding the edge $u_0 v_0$.

**Theorem 6.4.7** If $G$ and $H$ both are $k$-critical graphs, then $G \nabla H$ is a $k$-critical graph.

In order for $G \nabla H$ to be a planar graph, we can only identify two pairs of vertices at most. Let $H_1$ be the Koester’s graph and $H_{i+1} = H_i \nabla H_1$ (identifying one edge in each graph). We have $|E(H_{i+1})| = |E(H_i)| + 78$ and $|V(H_{i+1})| = |V(H_i)| + 38$. Therefore $|E(H_n)| = 78(n - 1) + 80$ and $|V(H_n)| = 38(n - 1) + 40$. The edge density of $H_n$ is

$$\frac{78(n - 1) + 80}{38(n - 1) + 40} \rightarrow \frac{39}{19}$$

as $n \to \infty$.

6.4.2 The upper bounds of edge density of planar 4-critical graphs
Abbott and Zhou also showed that if $G$ is a planar 4-critical graph then $|E(G)| / |V(G)| \leq 11/4 = 2.75$.

**Proof:** Let $F_i$ be the number of regions that has $i$ edges in its boundary. Let $v, e, r$ be the number of vertices, edges, and regions in $G$ respectively. Since every edge is in the boundaries of two regions, we have

$$2e = \sum_{i \geq 3} iF_i = 3F_3 + \sum_{i \geq 4} iF_i$$

$$2e - 3F_3 = \sum_{i \geq 4} iF_i \geq 4 \sum_{i \geq 4} F_i$$

We have an upper bound for the number of regions that are not a triangle:

$$\sum_{i \geq 4} F_i \leq \frac{2e - 3F_3}{4}$$

Thus

$$r \leq F_3 + \frac{2e - 3F_3}{4},$$

$$r \leq \frac{e}{2} + \frac{F_3}{4}.$$ (6.8)

We need to estimate $F_3$, the number of triangles. We need the following lemma.
Lemma 6.4.8 It $G$ is a 4-critical graph, then either $G$ is an odd wheel or it does not contain any wheels.

Proof: Suppose that $G$ is not an odd wheel. Obviously $G$ cannot contain an odd wheel. Suppose that $G$ contains an even wheel with center $w$ and $v_1, v_2, ..., v_{2l}$ on the rim. Since $G$ is 4-critical, $G - v_1v_{2l}$ can be 3-coloured. Without loss of generality, we assume that $w$ is coloured with colour 3 and $v_1$ is coloured with colour 1. Then $v_2$ must be coloured with colour 2. $v_3$ must be coloured with colour 1. $v_4$ must be coloured with colour 2....$v_{2l}$ must be coloured with colour 2. However, this is a 3-colouring of $G$ as well. This contradicts to the assumption that $\chi(G) = 4$. \[ \square \]

If $G$ is an odd wheel, the edge density of $G$ is less than 2. We assume that $G$ is not an odd wheel. Therefore, every vertex in $G$ must be incident with a region that is bounded by at least 4 edges. Since every region bounded by $i$ edges has $i$ vertices in its boundary, we have

$$\sum_{i\geq 4} iF_i \geq v.$$ 

Using (6.7), we have

$$2e = \sum_{i\geq 3} iF_i = 3F_3 + \sum_{i\geq 4} iF_i \geq 3F_3 + v.$$ 

That gives us

$$F_3 \leq \frac{2e - v}{3}.$$ 

Substituting this into (6.8), we have

$$r \leq \frac{e}{2} + \frac{2e - v}{12} = \frac{2}{3}e - \frac{1}{12}v.$$ 

This and the Euler’s formula gives us

$$v + \left(\frac{2}{3}e - \frac{1}{12}v\right) - e \geq 2$$

$$\frac{11}{12}v - \frac{e}{3} \geq 2.$$ 

That gives

$$\frac{e}{v} \leq \frac{11}{4} = 2.75.$$ 

\[ \square \]

Grunbaum raised four questions in his paper. We were able to answer three of them. The one question we could not answer was:
Problem 6.4.9 Does every planar 4-critical graph G satisfy $\delta(G) \leq 4$ where $\delta(G)$ is the minimum degree of G.

This is a modification of Dirac’s conjecture. Recall that we already know that for a planar graph $G$, $\delta(G) \leq 5$. Later, Abbot, Katchalski and Zhou proved that for every planar 4-critical graph $G$, $\delta(G) \leq 4$. Koester proved that for such graphs, the edge density $\frac{e}{n} < 2.5$ therefore $\delta(G) < 5$. In his proof, Koester used a result of Stiebitz (1987): If $G$ is a 4-critical graph, then $G$ contains at most $v$ triangles. Indeed, the condition $F_3 \leq v$ together with (6.8) gives us

$$r \leq \frac{e}{2} + \frac{v}{4}.$$

Using the Euler’s formula, we have

$$v + \frac{e}{2} + \frac{v}{4} - e \geq 2$$
$$\frac{5v}{4} \geq \frac{e}{2} + 2$$
$$e < \frac{5}{2}v.$$

Stiebitz’s result is a special case of another conjecture of Gallai:

Conjecture 6.4.10 (Gallai) If $G$ is a k-critical graph of n vertices then $G$ contains at most n complete subgraphs of order k − 1.

Theorem 6.4.11 (Stiebitz) If $G$ is a 4-critical graph of n vertices then $G$ contains at most n triangles.

Proof: If $G$ is an odd wheel, then the number of triangles is $n - 1$. So we assume that $G$ is not an odd wheel. Let the vertices of $G$ be $v_1, v_2, ..., v_n$ and the triangles of $G$ be $T_1, T_2, ..., T_t$. We define a vector $T^i = [T_1^i, T_2^i, ..., T_n^i]$ for each $T^i$ where

$$T^i_j = \begin{cases} 1 & \text{if } v_j \in T^i \\ 0 & \text{otherwise.} \end{cases}$$

We want to show that on the field of $\mathbb{Z}_2$, the set of vectors $\{T^i : i = 1, 2, ..., t\}$ is linearly independent. Assume that it is linearly dependent. Then without loss of generality, we may assume that for some positive integer $k$,

$$T^1 + T^2 + \cdots + T^k = 0.$$

---


That means that every vertex is contained in even number of triangles \( T^1, T^2, \ldots, T^k \) (\(*\)). Since \( G \) does not contain a wheel, there is an edge that is contained in only one of the triangles. Say that \( v_1v_2 \) is contained only in the triangle \( T^1 = \{v_1, v_2, v_3\} \). \( G - v_1v_2 \) can be coloured with 3 colours. Suppose that in this 3-colouring, both \( v_1 \) and \( v_2 \) are coloured with colour 1, \( v_3 \) is coloured with colour 3. Then every triangle except \( T^1 \) has one vertex of each colour and \( T^1 \) has two vertices of colour one and one vertex of colour 3. The sum of the entries in the columns corresponding to the vertices of colour 3 is \( k \) and the sum of the entries in the columns corresponding to the vertices of colour 1 is \( k + 1 \). If \( k \) is odd, then one of the vertices with colour 3 must be in odd number of the triangles; if \( k \) is even then \( k + 1 \) is odd and one of the vertices with colour 1 must be in odd number of the triangles. Either way, we have contradiction to (\(*\)). This proves the set of vectors \( \{T^j : j = 1, 2, \ldots, t\} \) is linearly independent. Therefore, \( t \leq n \).

Adapting Stiebitz’s technique, Abbott and Zhou\(††\) proved Gallai’s conjecture completely. We also showed that a 4-critical graph on \( n \)-vertices can have at most \( n - 1 \) complete \((k - 1)\)-subgraphs. If \( G \) is not an odd wheel, then \( G \) can contain at most \( n - 2 \) such subgraphs.