Chapter 6
GRAPH COLORING

A $k$-coloring of (the vertex set of) a graph $G$ is a function $c : V(G) \rightarrow \{1, 2, \ldots, k\}$ such that $c(u) \neq c(v)$ whenever $u$ is adjacent to $v$. If a $k$-coloring of $G$ exists, then $G$ is called $k$-colorable. The chromatic number of $G$ is defined as

$$\chi(G) = \min \{k : G \text{ is } k\text{-colorable}\}.$$ 

When $\chi(G) = k$, $G$ is called $k$-chromatic.

6.1 Lower bounds of the chromatic number

Let $\omega(G)$ be the clique number of $G$, that is, the maximal number of vertices in a complete subgraph of $G$. Obviously, we have

$$\chi(G) \geq \omega(G) \tag{6.1}$$

Given a $k$-coloring of $G$, the vertices being colored with the same color form an independent set. Let $G$ be a graph with $n$ vertices and $c$ a $k$-coloring of $G$. We define

$$V_i = \{v : c(v) = i\}$$

for $i = 1, 2, \ldots, k$. Each $V_i$ is an independent set. Let $\alpha(G)$ be the independence number of $G$, we have

$$|V_i| \leq \alpha(G).$$

Since

$$n = |V_1| + |V_2| + \cdots + |V_k| \leq k \cdot \alpha(G) = \chi(G) \alpha(G),$$

we have

$$\chi(G) \geq \frac{n}{\alpha(G)} \tag{6.2}.$$ 

These bounds can be very far off. For (6.2), there are graphs with $\frac{n}{\alpha(G)} < 2$ but $\chi(G)$ arbitrarily large.

(6.1) is not much better either. There are graphs with $\chi(G) > \omega(G)$, for example, $C_5$. $\chi(C_5) = 3$ and $\omega(C_5) = 2$. Let $G$ and $H$ be two graphs. We define the join of $G$ and $H$, $G \oplus H$ to be the graph obtained by joining every vertex in $G$ with
every vertex in $H$. Clearly, $\chi (G \oplus H) = \chi (G) + \chi (H)$ and $\omega (G \oplus H) = \omega (G) + \omega (H)$. So $\chi (C_5 \oplus C_5) = 6$ and $\omega (C_5 \oplus C_5) = 4$. Let $G_1 = C_5 \oplus C_5$ and $G_p = G_{p-1} \oplus G_{p-1}$, we have

$$\chi (G_1) = 6, \omega (G_1) = 4$$
$$\chi (G_2) = 12, \omega (G_2) = 8$$
$$\chi (G_3) = 24, \omega (G_3) = 16$$
$$\cdots$$
$$\chi (G_p) = 3 \cdot 2^p, \omega (G_p) = 2^{p+1}$$

$$\chi (G_p) - \omega (G_p) = 2^p$$

The difference between $\chi (G)$ and $\omega (G)$ can be arbitrarily large. Furthermore, even when $\omega (G) = 2$, we can have $\chi (G)$ arbitrarily large.

When $\omega (G) = 2$, $G$ is called triangle-free.

### 6.1.1 Tutte’s construction of triangle-free $k$-chromatic graphs

We construct $G_k$ recursively.

$G_2 = K_2$.

Let $n_k = |G_k|$, the number of vertices in $G_k$.

Suppose that $G_{k-1}$ has been built. Let $I_k$ be an independent set of $(n_{k-1} - 1) (k - 1) + 1$ vertices. For every $(k - 1)$-subset of $I_k$, we take a copy of $G_{k-1}$ and match the vertices in that copy to the vertices in the $(k - 1)$-subsets. The result is $G_k$.

**Example:** $G_3$.

**Claim 6.1.1** $G_k$ is triangle-free.

**Proof.** Prove by induction on $k$.

$G_2$ is obviously triangle-free. Assume that $G_{k-1}$ is triangle-free. No vertex in $I_k$ can be in a triangle: Let $x$ be a vertex in $I_k$. All vertices that are adjacent to $x$ are in different copies of $G_{k-1}$. So no two neighbors of $x$ are adjacent. Since there are no triangles in the copies of $G_{k-1}$, there is no triangle in $G_k$.

**Claim 6.1.2** $\chi (G) = k$.

**Proof.** By induction on $k$. Since it is easy to see that $G_k$ is $k$-colorable (color all the copies of $G_{k-1}$ with colors $1, 2, \ldots, k - 1$ and color the vertices in $I_k$ with color $k$), we only need to show that $G_k$ is not $(k - 1)$-colorable. Suppose that $G_k$ is $(k - 1)$-colorable and $c$ is a $(k - 1)$-coloring of $G_k$. By the pigeonhole principle, there are at least $n_{k-1}$ vertices that are colored with the same color. Then the copy of $G_{k-1}$ must be colored with $k - 2$ colors. Contradiction.
6.2 Upper bounds

6.2.1 The greedy algorithm

Let the vertices of a graph $G$ be $v_1, v_2, ..., v_n$ and the maximum degree of $G$ be $\Delta$. For $i = 1, 2, ..., n$ we color $v_i$ with the smallest number that is available, i.e. the number that has not been used on the vertices in $\{v_1, v_2, ..., v_{i-1}\}$ that are adjacent to $v_i$. Since there are at most $\Delta$ vertices that are adjacent to $v_i$, this algorithm will not use more than $\Delta + 1$ colors. Therefore we have

**Theorem 6.2.1**

$$\chi(G) \leq \Delta(G) + 1$$  \hspace{1cm} (6.3)

(6.3) can be pretty bad. The graph $K_{1,p}$ is called a star. It has $\Delta(G) = p$ and $\chi(G) = 2$. However, for a complete graph $K_n$ and an odd cycle, we have $\chi(G) = \Delta(G) + 1$. Therefore, (6.3) cannot be improved in general.

**Theorem 6.2.2** Suppose that in every subgraph $H$ of $G$ there is a vertex with degree at most $\delta$ in $H$.

$$\chi(G) \leq \delta + 1$$

**Proof.** There is a vertex with degree at most $\delta$ in $G$. Label that vertex $v_n$. There is a vertex with degree at most $\delta$ in $G - v_n$. Label that vertex $v_{n-1}$. Label the vertex with degree at most $\delta$ in $G - \{v_n, v_{n-1}\} v_{n-2}$. Continue to label all the vertices in $G$ as $v_1, v_2, ..., v_n$. Apply the greedy algorithm according to this labeling. At each step, the vertex we are going to color is adjacent to at most $\delta$ vertices that are already colored. Therefore, $\delta + 1$ colors will be enough. \[\square\]

For a star $K_{1,p}$ Theorem 6.2.1 provides the upper bound $\chi(G) \leq p + 1$. Since every subgraph of $K_{1,p}$ has a vertex of degree at most one, Theorem 6.2.2 gives the upper bound $\chi(G) \leq 2$.

**Definition 6.2.3** A graph is called a wheel ($W_n$) if it is obtained by joining a new vertex to every vertex in a cycle $C_n$. The wheel is called odd (or even) if $n$ is odd (or even).

In the figure that follows is the wheel $W_7$. It has maximum degree 7. All its subgraphs contain a vertex of degree at most three. Theorem 6.2.1 gives an upper bound $\chi(G) \leq 8$ and Theorem 6.2.2 gives $\chi(G) \leq 4$. Actually we can see that $\chi(G) = 4$.
6.2.2 Brook’s Theorem

Theorem 6.2.2 is difficult to apply in general. Another way to improve (6.3) is the following theorem.

**Theorem 6.2.4 (Brook’s Theorem)** Suppose that \( \Delta(G) = k \geq 2 \). \( \chi(G) \leq k \) unless \( G \) contains a \( K_{k+1} \) or \( k = 2 \) and \( G \) contains an odd cycle.

Before we prove Brook’s Theorem, we need some definitions about the connectivity of a graph. We denote the graph obtained by deleting vertices \( v_1, v_2, ..., v_k \) (and all the edges that are incident to at least one of these vertices) from \( G \) as \( G - \{v_1, v_2, ..., v_k\} \) and we write \( G - \{v\} \) as \( G - v \). A vertex in \( G \) is called a cut-vertex if \( G - v \) has more components than \( G \). \( G \) is called 2-connected if \( G \) is connected and \( G \) does not contain any cut-vertex. (That is, you need to remove at least two vertices to disconnect the graph.) Similarly we say that \( G \) is \( k \)-connected if \( G \) is connected and one needs to remove at least \( k \) vertices to disconnect the graph. A maximal subgraph of \( G \) that is 2-connected is called a block. We can construct a block-graph \( B(G) \) for a graph \( G \): The vertices of \( B(G) \) are the blocks and cut-vertices of \( G \). There is an edge joining a cut-vertex with a block if the cut-vertex is in that block. In the following figure, a graph \( G \) and its block graph \( B(G) \) is shown.

![Graph and Block Graph](image)

\( B(G) \) is always a tree. (Why?) If a block is a leaf in \( B(G) \), we say that block is an end-block.

**Proof of Brook’s Theorem:** The theorem is true for \( k = 2 \): If \( G \) does not contain any odd cycles, then \( G \) is bipartite therefore \( \chi(G) \leq 2 \). Let \( k > 3 \) be the first value of \( \Delta(G) \) such that the theorem is not true and let \( G \) be a counter example with the least number of vertices. That is, \( \Delta(G) = k \), \( \chi(G) > k \) and every graph with less vertices than \( G \) that has maximum degree \( k \) can be coloured with \( k \) colours.

Claim 1: \( G \) must be connected. Then either one of the components of \( G \) is a smaller counter example or all its components can be coloured with \( k \) colours therefore \( G \) can be coloured with \( k \) colours. Either way, there is a contradiction.

Claim 2: \( G \) must be 2-connected. Suppose that \( G \) contains a cut-vertex \( u \). Each block of \( G \) can be coloured with \( k \) colours. Then we can combine those \( k \)-colourings together to obtain a \( k \)-colouring of \( G \). This is a contradiction.
Claim 3: \( G \) must contain three vertices \( u_1, u_2 \) and \( v \) such that \( u_1 \) is not adjacent to \( u_2 \); both \( u_1 \) and \( u_2 \) are adjacent to \( v \) and \( G - \{u_1, u_2\} \) is connected.

Case 1: \( G \) is 3-connected. Since \( G \) is not a completed graph, there are two vertices \( x \) and \( y \) such that \( x \) is not adjacent to \( y \). Let the shortest path from \( x \) to \( y \) be \( x = v_0, v_1, ..., v_p = y \) and \( p \geq 2 \). Since this path is the shortest, \( v_0 \) is not adjacent to \( v_2 \). Therefore, we can choose \( u_1 = v_0, u_2 = v_2 \) and \( v = v_1 \).

Case 2: \( G \) is 2-connected but not 3-connected. There exist two vertices \( u \) and \( v \) such that \( G - \{u, v\} \) is disconnected. Then \( G - v \) contains cut-vertices (for example, the vertex \( u \)). However, there is no cut-vertex in \( G \). So \( v \) must be adjacent to at least one vertex in every block of \( G - v \). Let \( u_1 \) and \( u_2 \) be two vertices in two different end-blocks of \( G - v \) that are adjacent to \( v \). The triple \( u_1, u_2 \) and \( v \) satisfy the conditions of Claim 3.

Then we order the vertices of \( G \) in such a way that \( v_1 = u_1, v_2 = u_2 \) and \( v_n = v \). \( v_3, v_4, ..., v_{n-1} \) are ordered in such a way that if \( i < j \) then the distance from \( v \) to \( v_i \) is greater than or equal to the distance from \( v \) to \( v_j \) for all \( 3 \leq i, j \leq n - 1 \). When we apply the greedy algorithm to \( G \) according to this ordering, \( v_1 \) and \( v_2 \) will be coloured with colour 1. When we are colouring \( v_i \) \((i < n)\), there is at least one neighbor of \( v_1 \) that has not been coloured. Therefore, there is at least one colour available for \( v_1 \) if we have \( k = \Delta(G) \) colours. When we are colouring the last vertex \( v_n = v \), at most \( \Delta - 1 \) colours were used on the neighbors of \( v \) (since \( v_1 \) and \( v_2 \) are coloured with the same colour), and there is at least one colour for \( v \). This shows that \( G \) can be coloured with \( k \) colours as well.

\( \square \)

### 6.3 Planar graphs and the Four Colour Theorem

One earlier focus of graph theory is the Four Colour Problem:

Given a map of countries such that every country is a continuous piece, is it always possible to use four colors to color the countries such that any two countries with a common boundary are colored with different colors?

It is easy to see that three colors would not be enough and no one had found an example of map that needed more than four colors. However, no one can prove that such examples do not exist for a long time. It is until 1976 with the help of thousands hours of computer time, a proof was found.

#### 6.3.1 Dual graphs

A map can be represented by a graph such that each country is represented by a vertex and two countries share a boundary if and only if the corresponding vertices are adjacent. That graph is called a dual graph of the map. A coloring of the map satisfying the condition of the four color problem corresponding to a coloring of the vertices of the dual graph. A graph is planar if it can be drawn (or embedded) in the plane such that no two edges crossing each other. The four color problem is equivalent to

If \( G \) is a planar graph, is it true that \( \chi(G) \leq 4 \)?
6.3.2 Euler’s formula

We first study some properties of the planar graphs. The most important property is the Euler’s formula:

Let $G$ be a connected planar graph and $v, e, r$ the numbers of vertices, edges and regions of $G$ respectively.

$$v - e + r = 2.$$  \hspace{1cm} (6.4)

The formula is easy to prove by induction on $v$.

Since every region is bounded by at least three edges and an edge can be used twice in the boundaries, we have

$$3r \leq 2e$$

or

$$r \leq \frac{2e}{3}.$$  

Substitute this into (6.4), we have

$$v - e + \frac{2e}{3} \geq 2$$

$$v \geq \frac{1}{3}e + 2$$

$$e \leq 3v - 6.$$  \hspace{1cm} (6.5)

Corollary 6.3.1 $K_5$ is not planar.

Corollary 6.3.2 $K_{3,3}$ is not planar.

(6.5) cannot be used directly to prove Corollary 6.3.2. We need a stronger version of (6.5) for bipartite graphs. Since a bipartite graph does not contain any triangles, a region is bounded by at least 4 edges. For all bipartite graphs, we have

$$4r \leq 2e$$

$$r \leq \frac{1}{2}e.$$  

That will imply

$$v - e + \frac{1}{2}e \geq 2$$

$$v \geq \frac{1}{2}e + 2$$

$$e \leq 2v - 4.$$  \hspace{1cm} (6.6)

(6.6) implies Corollary 6.3.2.

Subdividing an edge or performing an elementary subdivision of an edge $uv$ of a graph $G$ is the operation of deleting $uv$ and adding a path $u, w, v$ through a
new vertex \( w \). A \textit{subdivision} of \( G \) is a graph obtained from \( G \) by a sequence of elementary subdivisions, turning edges into paths through new vertices of degree 2. Subdividing edges does not affect planarity. Therefore any subdivision of \( K_5 \) or \( K_{3,6} \) is not planar either. Kuratowski [1930] proved that \( G \) is planar if and only if \( G \) contains no subdivision of \( K_5 \) or \( K_{3,3} \). However, this condition is not easy to check for large graphs.

6.3.3 Planarity testing

There are more efficient algorithms to test whether a graph \( G \) is planar. The output of these algorithm is either a drawing of \( G \) in the plane (when \( G \) is planar) or the statement that \( G \) is not planar. These algorithm all use the concept of \( H \)-bridge.

If \( H \) is a subgraph of \( G \), and \( H \)-bridge of \( G \) is either (1) an edge not in \( H \) whose endpoint are in \( H \) or (2) a component of \( G - V(H) \) together with the edges (and vertices of attachment) that connect it to \( H \). The \( H \)-bridges are the “pieces” that must be added to an embedding of \( H \) to obtain an embedding of \( G \).

Given a cycle \( H \) in a graph \( G \), two \( H \)-bridges \( A, B \) conflict if they have three common vertices of attachment or if there are four vertices \( v_1, v_2, v_3, v_4 \) in cyclic order on \( H \) such that \( v_1, v_3 \) are vertices of attachment of \( A \) and \( v_2, v_4 \) are vertices of attachment of \( B \). In the latter case, we say the \( H \)-bridges cross. The \textit{conflict graph} of \( H \) is a graph whose vertices are the \( H \)-bridges of \( G \), with conflicting \( H \)-bridges adjacent.

\textbf{Theorem 6.3.3} (Tutte 1958) A graph \( G \) is planar if and only if for every cycle \( C \) in \( G \), the conflict graph of \( C \) is bipartite.

It is easy to see that why for a planar graph \( G \), the conflict graph must be bipartite. In a planar embedding of \( G \), all the \( H \)-bridges are either inside of \( C \) or outside of \( C \). The ones on the same side cannot be conflicting.

\textbf{Example 6.3.4} The conflict graph of a \( C_5 \) in \( K_5 \).

\begin{center}
\begin{tikzpicture}
  \draw (0,0) -- (1,1) -- (2,0) -- (3,1) -- (0,0);
  \draw (0,0) -- (1,1);
  \draw (1,1) -- (2,0);
  \draw (2,0) -- (3,1);
  \draw (3,1) -- (0,0);
  \filldraw[black] (0,0) circle (2pt);
  \filldraw[black] (1,1) circle (2pt);
  \filldraw[black] (2,0) circle (2pt);
  \filldraw[black] (3,1) circle (2pt);

  \node at (-1,1) {\textit{K}_5};
  \node at (0.5,0.5) {1}
  \node at (-0.5,-0.5) {2}
  \node at (1.5,-0.5) {3}
  \node at (0.5,1.5) {4}

  \draw (4,0) -- (5,1) -- (3,0) -- (2,1) -- (4,0);
  \draw (4,0) -- (5,1);
  \draw (5,1) -- (3,0);
  \draw (3,0) -- (2,1);
  \draw (2,1) -- (4,0);
  \filldraw[black] (4,0) circle (2pt);
  \filldraw[black] (5,1) circle (2pt);
  \filldraw[black] (3,0) circle (2pt);
  \filldraw[black] (2,1) circle (2pt);

  \node at (0.5,0.5) {1}
  \node at (-0.5,-0.5) {2}
  \node at (1.5,-0.5) {3}
  \node at (0.5,1.5) {4}

  \node at (2,-1) {The conflicting graph.}
\end{tikzpicture}
\end{center}
Example 6.3.5 The conflict graph of a $C_6$ in $K_{6,6}$.

There are linear-time planarity-testing algorithms due to Hopcroft and Tarjan (1974) and to Booth and Luecker (1976), but these are very complicated. An earlier algorithm due to Demoucron, Malgrange and Pertuiset (1964) is much simpler. It is not linear but runs in polynomial time. The idea is that if a planar embedding of $H$ can be extended to a planar embedding of $G$, then in that extension every $H$-bridge of $G$ appears inside a single face of $H$. The algorithm builds increasingly larger plane subgraphs $H$ of $G$ that can be extended to an embedding of $G$ if $G$ is planar. We want to enlarge $H$ by making small decisions that won’t lead to trouble.

The basic enlargement step is the following. (1) Choose a face $F$ that can accept an $H$-bridge $B$; a necessary condition for $F$ to accept $B$ is that its boundary contain all vertices of attachment of $B$. (2) Although we do not know the best way to embed $B$ in $F$, each particular path in $B$ between vertices of attachment by itself has only one way to be added across $F$, so we add a single such path. (3) If there is an $H$-bridge that is not acceptable to any face of $H$, then $G$ is not planar.

Example 6.3.6

The first subgraph $H$ used is the cycle 1, 2, 3, 4, 5. Then the path 3, 7, 6, 5....
Example 6.3.7

For the cycle $1, 2, 3, 4, 8, 7, 6, 5$, the edges 14, 27, 36 are all crossing. The graph is nonplanar.

6.3.4 The chromatic number of a planar graph

(6.5) implies that for a planar graph $G$, the degree sum has the upper bound

$$\sum_v \deg(v) = 2e \leq 6v - 12.$$ 

There is a vertex with degree at most 5. Since every subgraph of a planar graph is also a planar graph, every subgraph of $G$ has a vertex with degree at most 5. By Theorem 6.2.2, $\chi(G) \leq 6$. So the six color theorem is easy to proof. With a little of more work, we can prove the five color theorem:

Theorem 6.3.8 Every planar graph $G$ has chromatic number at most five.

Proof. Suppose that the theorem is not true. Let $G$ be the smallest planar graph (in terms of number of vertices) that cannot be colored with five colors. Let $v$ be a vertex in $G$ that has the minimum degree. We know that $\deg(v) \leq 5$.

Case 1: $\deg(v) \leq 4$. $G - v$ can be colored with five colors. There are at most colors have been used on the neighbors of $v$. There is at one color available for $v$. So $G$ can be colored with five colors, a contradiction.

Case 2: $\deg(v) = 5$. $G - v$ can be colored with five colors. If two of the neighbors of $v$ are colored with the same color, then there is a color available for $v$. So we may assume that all the vertices that are adjacent to $v$ are colored with colors 1, 2, 3, 4, 5 in the clockwise order. We will call these vertices $v_1, v_2, ..., v_5$. Consider all the vertices being colored with colors 1 or 3 (and all the edges among them). If this subgraph $G$ is disconnect and $v_1$ and $v_3$ are in different components, then we can switch the colors 1 and 3 in the component with $v_1$. This will still be a 5-coloring of $G - v$. Furthermore, $v_1$ is colored with color 3 in this new 5-coloring and $v_3$ is still colored with color 3. Color 1 would be available for $v$, a contradiction. Therefore $v_1$ and $v_3$ must be in the same component in that subgraph, i.e., there is a path from $v_1$ to $v_3$ such that every vertices on this path is colored with either color 1 or color 3.
Similarly, there is a path from $v_2$ to $v_4$ such that every vertices on that path has either color 2 or color 4. These two paths must intersect even though they have no vertex in common. Then there must be two edges crossing each other. This contradiction concludes the proof.